Goursat Systems of $\mathbb{R}^n$ Satisfying in Each Dimension a Singularity of Order Two

Mohamad H. Cheaito and Hassan Zeineddine

Lebanese University, Faculty of Sciences
Department of Mathematics, Hadath, Lebanon
mohamad_cheaito@yahoo.com
zeineddine_hassan@hotmail.com

Abstract
In this article we study the 2-distributions in $\mathbb{R}^n$ satisfying everywhere the Goursat condition. We give the normal form at a neighborhood of a point where the small growth vector has a singularity of order 2 at any step. After the Goursat case studied by Kumpera-Ruiz and Murray, this article is the most natural model.

Mathematics Subject Classifications: Primary 58A30, 58F36; Secondary 58E10, 58B20

Keywords: Non holonomic distribution, normal forms

Introduction:
Let $E$ be a 2-distribution on $\mathbb{R}^n$. We denote by

$$E^1 = E_1 = E, \quad E^i = [E^{i-1}, E^{i-1}] \text{ and } E_i = [E, E_{i-1}]$$

A small growth vector (sgv) of $E$, at a point $p \in \mathbb{R}^n$, is the sequence

$$[r_1(p), r_2(p), \cdots]_S$$

where $r_i(p) = \dim E_i(p)$, for every $i \geq 1$.

The great growth vector, at $p$, is the sequence

$$[m_1(p), m_2(p), \cdots]_G$$

where $m_j(p) = \dim E^j(p)$, for every $j \geq 1$.

If the dimensions of $E_i$ (resp. $E^j$) are independent of $p$, the distribution is called regular (resp. totally regular).
If the great growth vector, at a point \( p \in \mathbb{R}^n \), is \([2, 3, 4, \cdots, n]_G\), the distribution is called *distribution satisfying the Goursat condition at p*. Moreover, if \( E \) satisfies, on a neighborhood of \( p \), the Goursat condition its annihilator \( E^\perp \), is called *Goursat system* and denoted by (GS).

The classification of the distributions, with respect to the small and great growth vectors, was the object of many articles. The beginning was by Engel [3], Where he gave the normal form of the (GS) in dimension 4.

In an article written in 1910, E. Cartan [1] certified that, if a 2-distribution satisfies th Goursat condition, it has a clear normal form called the (GNF), it generalized the normal form given by Engel. In 1978 Giaro, Kumpe and Ruiz proved that the result is false after the dimension 5 [4]. In a such case it is presented 2 non equivalent models. In 1981, [5], Kumpe and Ruiz gave the different normal forms in dimension \( n \leq 6 \).

The classification, of models, in dimension 7 and 8 are given by [2]. The study of the models in dimension \( n \) is also open. We say, [7], when the small and the great growth vector are the same we have the system (GNF).

Zhitomirskii [8] gave the asymptotic normal forms of the regular distributions and the generic case studied in many articles for example [9].

In this article we consider \( n \) is arbitrary. Using the normal form given in [2], we obtain the model, satisfying at a neighborhood of a point the small growth vector

\[ [2, 3, 4, 5, 5, \cdots, n-1, n-1, n]_S \]

In fact, we prove the following

**Theorem 1** Let \( E \) be the Goursat system, Satisfying at a point \( p \in \mathbb{R}^n \) the small growth vector \([2, 3, 4, 5, 5, \cdots, n-1, n-1, n]_S \). Then there exists a local system of coordinates \((x, U)\), around \( p \), such that \( E \) is spanned by the vectors

\[
\begin{align*}
  v_1 &= \frac{\partial}{\partial x_n} \\
  v_2 &= -x_n \frac{\partial}{\partial x_1} + x_n x_3 \frac{\partial}{\partial x_2} + x_n x_4 \frac{\partial}{\partial x_3} + \cdots + x_n x_{n-1} \frac{\partial}{\partial x_{n-2}} + \frac{\partial}{\partial x_{n-1}}
\end{align*}
\]

The *Goursat systems* are given by the following theorem:

**Theorem 2** ([2], [5]) Let \( E \) be a 2-distribution on \( \mathbb{R}^n \), satisfying in each point, the condition of Goursat then

\[
E^\perp = \left\{ \begin{array}{l}
\omega_1 = dx_2 + x_3 dx_1 \\
\omega_2 = dx_3 + x_4 dx_1 \\
\omega_3 = dx_{i_3} + x_5 dx_{j_3}, (i_3, j_3) \in \{(4, 1), (1, 4)\} \\
\omega_4 = dx_{i_4} + X_6 dx_{j_4}, (i_4, j_4) \in \{(5, j_3), (j_3, 5)\} \\
\vdots \\
\omega_{n-2} = dx_{i_{n-2}} + X_n dx_{j_{n-2}}, (i_{n-2}, j_{n-2}) \in \{(n-1, j_{n-3}), (j_{n-3}, n-1)\}
\end{array} \right.
\]
Where
\[ X_l = \begin{cases} 
  x_l, & \text{if } (i_{l-2}, j_{l-2}) = (j_{l-3}, l-1) \\
  x_l + c_l, & \text{if } (i_{l-2}, j_{l-2}) = (l-1, j_{l-3})
\end{cases} \]

for \( 6 \leq l \leq n \) and \( c_6, c_7, \ldots, c_{n-2} \) are real arbitrary constants. ♦

This theorem gives the different Goursat systems denoted by (GS).

**Normal form of the Goursat systems presenting in each dimension a singularity of order 2.**

**Definition 3** Let \( S \) be a Goursat system. \( S \) is called presenting a transposition of order \( l \), \( l \in \{3, 4, \ldots, n-2\} \), if
\[
\begin{align*}
\omega_{l-1} &= dx_{i_{l-1}} + X_{l+1} dx_{j_{l-1}} \\
\omega_l &= dx_{j_{l-1}} + x_{l+2} dx_{l+1}
\end{align*}
\]

**Definition 4** If the small growth vector of a 2-distribution \( E \) on \( \mathbb{R}^n \), at a point \( p \) of \( \mathbb{R}^n \), has the form \([2, 3, \cdots, s, s, \cdots, s, \cdots, n]\) (denoted by \([2, 3, \cdots, s_k, \cdots, n]\))
\[ \text{k times} \]
the distribution is called a distribution presenting, in the dimension \( s \), a singularity of order \( k \).

**Remark**: If the distribution satisfies the condition of Goursat the dimensions 2, 3 and \( n \) are of order 1 at every point.

**Notation**: The system of Goursat satisfying, at every point \( x \in \mathbb{R}^n \), the condition \([2, 3, 4_k, 5_k, \cdots, (n-1)_k, n]\) \( S \) is denoted by \((GS_k)\)

**Main Result**

**Theorem 5** Let \( E \) be a 2-distribution on \( \mathbb{R}^n \), satisfying at every point the Goursat condition, such that at \( x_0 \in \mathbb{R}^n \) we have \([2, 3, 4_2, 5_2, \cdots, (n-1)_2, n]\) \( S \).
Then there exists a local system of coordinates \((x, U)\), around \( x_0 \), such that
\[ E^\perp = \begin{cases} 
\omega_1 &= dx_2 + x_3 dx_1 \\
\omega_2 &= dx_3 + x_4 dx_1 \\
\omega_3 &= dx_4 + x_5 dx_1 \\
\omega_4 &= dx_5 + x_6 dx_1 \\
&\vdots
\omega_{n-3} &= dx_{n-2} + x_{n-1} dx_1 \\
\omega_{n-2} &= dx_1 + x_n dx_{n-1}
\end{cases} \]

It means, \( E \) is spanned by
\[ v_1 = \frac{\partial}{\partial x_1}, \text{ and } v_2 = -x_n \frac{\partial}{\partial x_1} + x_3 \frac{\partial}{\partial x_2} + x_4 \frac{\partial}{\partial x_3} + \cdots + x_{n-1} \frac{\partial}{\partial x_{n-2}} + \frac{\partial}{\partial x_{n-1}} \]

**Proof:** Suppose that \( x_0 = 0 \). In the proof X means a non calculated vector field. \( E \) satisfies the Goursat condition, by theorem 2 we obtain the expression of \( E^\perp \).

The property is true for \( n = 5, 6, 7, 8 \).

Suppose that \( n \geq 9 \). We prove the property by 3 steps:
1) \( \omega_{n-2} = dx_{j_{n-2}} + x_n dx_{n-1} \) (i.e. \( E \) presents a transposition of order \( n-2 \))
2) There is no \( l \in \{ 3, 4, \ldots, n-3 \} \) such that \( \omega_l = dx_{j_{l+1}} + x_{l+2} dx_{l+1} \)
3) All the constants, of the Goursat systems, can be transformed to 0 by change of variables.

**Step 1:** By the expression of \( E \) we have 2 possibilities
i) \[ dx_{n-3} + X_{n-2} dx_{j_{n-5}} \]
\[ dx_{n-2} + X_{n-1} dx_{j_{n-5}} \]
\[ dx_{j_{n-5}} + x_n dx_{n-1} \]
\[ dx_{n-1} + X_n dx_{j_{n-5}} \]
\[ dx_{n-1} + X_{n-1} dx_{n-2} \]
\[ dx_{n-2} + x_n dx_{n-1} \]

and

ii) \[ dx_{j_{n-5}} + x_{n-2} dx_{n-3} \]
\[ dx_{n-2} + X_{n-1} dx_{n-3} \]
\[ dx_{n-3} + x_n dx_{n-1} \]
\[ dx_{n-1} + X_n dx_{n-3} \]
\[ dx_{n-1} + X_{n-1} dx_{n-2} \]
\[ dx_{n-2} + x_n dx_{n-1} \]

Consider the case i): a) and d) satisfy 1). Prove that the cases b) and d) are not possible.
case b): E is spanned by 
\[ v_1 = \frac{\partial}{\partial x_n}, \quad v_2 = p_1 \frac{\partial}{\partial x_1} + \cdots + \frac{\partial}{\partial x_{j-3}} + \cdots + p_{n-4} \frac{\partial}{\partial x_{n-4}} - X_{n-2} \frac{\partial}{\partial x_{n-3}} - X_{n-1} \frac{\partial}{\partial x_{n-2}} - X_n \frac{\partial}{\partial x_{n-1}} \]

where \( p_1 = p_1(x_5, x_6, \ldots, x_{n-3}), \ p_2 = p_2(x_3, x_4, \ldots, x_{n-3}) \) and \( p_j = p_j(x_{j+1}, x_{j+2}, \ldots, x_{n-3}) \) for every \( j \in \{3, 4, \ldots, n-4\} \), then we have:

\[
E_2 = \text{span}\left\{ \frac{\partial}{\partial x_n}, \frac{\partial}{\partial x_{n-1}}, v_2 \right\}
\]
\[
E_3 = \text{span}\left\{ \frac{\partial}{\partial x_n}, \frac{\partial}{\partial x_{n-1}}, \frac{\partial}{\partial x_{n-2}}, v_2 \right\}
\]
\[
E_4 = \text{span}\left\{ \frac{\partial}{\partial x_n}, \frac{\partial}{\partial x_{n-1}}, \frac{\partial}{\partial x_{n-2}}, \frac{\partial}{\partial x_{n-3}}, v_2 \right\}
\]

we deduce \( \text{dim}(E_4(0)) = 5 \), impossible.

case c): In this case E is spanned by 
\[ v_1 = \frac{\partial}{\partial x_n}, \text{ and } v_2 = x_{n-1}Z + \frac{\partial}{\partial x_{n-2}} - X_n \frac{\partial}{\partial x_{n-1}}, \text{ with } Z = p_1 \frac{\partial}{\partial x_1} + p_2 \frac{\partial}{\partial x_2} + \cdots - \frac{\partial}{\partial x_{j-5}} + \cdots + p_{n-4} \frac{\partial}{\partial x_{n-4}} + X_{n-2} \frac{\partial}{\partial x_{n-3}} \]

Then:

\[
E_2 = \text{span}\left\{ \frac{\partial}{\partial x_n}, \frac{\partial}{\partial x_{n-1}}, x_{n-1}Z + \frac{\partial}{\partial x_{n-2}} \right\}
\]
\[
E_3 = \text{span}\left\{ \frac{\partial}{\partial x_n}, \frac{\partial}{\partial x_{n-1}}, \frac{\partial}{\partial x_{n-2}}, Z \right\}
\]
\[
E_4 = \text{span}\left\{ \frac{\partial}{\partial x_n}, \frac{\partial}{\partial x_{n-1}}, \frac{\partial}{\partial x_{n-2}}, [v_2, Z] = \frac{\partial}{\partial x_{n-3}}, Z \right\}
\]

we obtain \( \text{dim}(E_4(0)) = 5 \), impossible.

Case ii): a) and d) satisfy 1).

For b)

E is spanned by \( v_1 = \frac{\partial}{\partial x_n}, \text{ and } v_2 = x_{n-2}X + \frac{\partial}{\partial x_{n-3}} - x_{n-2} \frac{\partial}{\partial x_{n-1}} - X_{n-1} \frac{\partial}{\partial x_{n-2}} - X_n \frac{\partial}{\partial x_{n-1}}, \text{ then we have:} \)
\[ E_2 = \text{span}\{ \frac{\partial}{\partial x_n}, \frac{\partial}{\partial x_{n-1}}, v_2 \} \]

\[ E_3 = \text{span}\{ \frac{\partial}{\partial x_n}, \frac{\partial}{\partial x_{n-1}}, \frac{\partial}{\partial x_{n-2}}, x_{n-2}(X + \frac{\partial}{\partial x_{j_{n-5}}}) + \frac{\partial}{\partial x_{n-3}} \} \]

\[ E_4 = \text{span}\{ \frac{\partial}{\partial x_n}, \frac{\partial}{\partial x_{n-1}}, \frac{\partial}{\partial x_{n-2}}, \frac{\partial}{\partial x_{n-3}}, X + \frac{\partial}{\partial x_{j_{n-5}}} \} \]

We obtain \( \dim(E_4(0)) = 5 \), impossible.

**case c)**

\( E \) is spanned by \( v_1 = \frac{\partial}{\partial x_n} \), and \( v_2 = x_{n-1}Z + \frac{\partial}{\partial x_{n-2}} - X_n \frac{\partial}{\partial x_{n-1}} \), with

\[ Z = x_{n-2}X - \frac{\partial}{\partial x_{n-3}} + x_{n-2} \frac{\partial}{\partial x_{j_{n-5}}} \]

then we have:

\[ E_2 = \text{span}\{ \frac{\partial}{\partial x_n}, \frac{\partial}{\partial x_{n-1}}, x_{n-1}Z + \frac{\partial}{\partial x_{n-2}} \} \]

\[ E_3 = \text{span}\{ \frac{\partial}{\partial x_n}, \frac{\partial}{\partial x_{n-1}}, \frac{\partial}{\partial x_{n-2}}, Z \} \]

\[ E_4 = \text{span}\{ \frac{\partial}{\partial x_n}, \frac{\partial}{\partial x_{n-1}}, \frac{\partial}{\partial x_{n-2}}, Z, [v_2, Z] = X + \frac{\partial}{\partial x_{j_{n-5}}} \} \]

we obtain \( \dim(E_4(0)) = 5 \), impossible.

**2nd Step:** Suppose that there exists, at least, \( \omega_i \), transposition of \( \omega_{i-1} \).

Let \( l \) the greatest integer such that \( \omega_{l-1} = dx_{i_{l-1}} + X_{l+1}dx_{j_{l-1}} \), where \( (i_{l-1}, j_{l-1}) \in \{(l, j_{l-1}), (j_{l-2}, l)\} \) and \( \omega_l = dx_{j_{l-2}} + X_{l+2}dx_{l+1} \).

\( l \) is the greatest integer such that \( \omega_l \) is a transposition, then \( \omega_{l+1} = dx_{l+2} + X_{l+3}dx_{l+1}, \ldots, \omega_{n-3} = dx_{n-2} + X_{n-1}dx_{l+1} \) and \( \omega_{n-2} = dx_{l+1} + X_n dx_{n-1} \), then \( E \) is spanned by

\[ v_1 = \frac{\partial}{\partial x_n}, v_2 = x_nZ + \frac{\partial}{\partial x_{n-1}} \]

where \( Z = x_{l+2}X + \frac{\partial}{\partial x_{l+1}} + X_{l+3} \frac{\partial}{\partial x_{l+2}} + \cdots + X_{n-1} \frac{\partial}{\partial x_{n-2}} \) and \( X = p_1 \frac{\partial}{\partial x_1} + p_2 \frac{\partial}{\partial x_2} + \cdots + p_l \frac{\partial}{\partial x_l} \)

with \( p_1 = p_l(x_5, x_6, \ldots, x_{l+1}), p_2 = p_2(x_3, x_5, \ldots, x_{l+1}) \) and \( p_j = p_j(x_{j+1}, x_{j+2}, \ldots, x_{l+1}) \), for every \( j \geq 3 \). We deduce:
P-1, prove that it is true for p.

\[ E_2 = \text{span}\{\frac{\partial}{\partial x_n}, \frac{\partial}{\partial x_{n-1}}, Z\} \]

\[ E_3 = \text{span}\{\frac{\partial}{\partial x_n}, \frac{\partial}{\partial x_{n-1}}, \frac{\partial}{\partial x_{n-2}}, Z |_{R^{n-3}}\} \]

\[ E_4 = \text{span}\{\frac{\partial}{\partial x_n}, \frac{\partial}{\partial x_{n-1}}, \frac{\partial}{\partial x_{n-2}}, x_n \frac{\partial}{\partial x_{n-3}}, Z |_{R^{n-3}}\} \]

where \( Z |_{R^t} = x_{t+2} + x_{t+3} \frac{\partial}{\partial x_{t+4}} + \cdots + x_{n+1} \frac{\partial}{\partial x_t}, \)
for every \( t \geq l + 2 \).

Prove, by induction on \( p \) (\( p \in \{1, 2, \cdots, \lceil \frac{n-l-2}{2} \rceil \})::

\[ E_{2p} = \text{span}\{(\frac{\partial}{\partial x_n}, \cdots, \frac{\partial}{\partial x_{n-p}}, x_n \frac{\partial}{\partial x_{n-p+1}}, x_n \frac{\partial}{\partial x_{n-p+2}}, \cdots, \}
\]

\[ E_{2p+1} = \text{span}\{(\frac{\partial}{\partial x_n}, \cdots, \frac{\partial}{\partial x_{n-p}}, x_n \frac{\partial}{\partial x_{n-p+1}}, x_n \frac{\partial}{\partial x_{n-p+2}}, x_n \frac{\partial}{\partial x_{n-p+3}}, \cdots, \}
\]

for \( p = 2 \)

\[ E_4 = \text{span}\{(\frac{\partial}{\partial x_n}, \frac{\partial}{\partial x_{n-1}}, \frac{\partial}{\partial x_{n-2}}, x_n \frac{\partial}{\partial x_{n-3}}, Z |_{R^{n-3}}\} \]

\[ E_5 = \text{span}\{(\frac{\partial}{\partial x_n}, \frac{\partial}{\partial x_{n-1}}, \frac{\partial}{\partial x_{n-2}}, x_n \frac{\partial}{\partial x_{n-3}}, x_n \frac{\partial}{\partial x_{n-4}}, Z |_{R^{n-4}}\} \]

Then the property is true for \( p = 2 \). Suppose that this property is true to the order \( p+1 \), prove that it is true for \( p \).

\[ E_{2p-1} = \text{span}\{(\frac{\partial}{\partial x_n}, \cdots, \frac{\partial}{\partial x_{n-p}}, x_n \frac{\partial}{\partial x_{n-p+1}}, x_n \frac{\partial}{\partial x_{n-p+2}}, x_n \frac{\partial}{\partial x_{n-p+3}}, \cdots, x_n \frac{\partial}{\partial x_{n-2p}}, Z |_{R^{n-p-1}}\} \]

Find the generators of \( E_{2p} \). We have, for every \( i \geq 0 \) and \( t \geq l + 3 \).

\[
[x_n^i \frac{\partial}{\partial x_t}, v_2] = x_n^{i+1} \frac{\partial}{\partial x_{t-1}}
\]

\[
[v_2, x_n^{2p-4} \frac{\partial}{\partial x_{n-2p+2}}] = x_n^{2p-3} [Z, \frac{\partial}{\partial x_{n-2p+2}}]
\]

\[
= x_n^{2p-3} \frac{\partial}{\partial x_{n-2p+1}}
\]

\[
[v_2, Z |_{R^{n-p-1}}] = x_n X_{n-p+1} \frac{\partial}{\partial x_{n-p-1}} \in E_{2p-1}
\]

Then
\[ E_{2p} = \text{span}\{ \frac{\partial}{\partial x_n}, \ldots, \frac{\partial}{\partial x_{n-p}}, x_n \frac{\partial}{\partial x_{n-p-1}}, x_n^2 \frac{\partial}{\partial x_{n-p-2}}, \ldots, x_n^{2p-3} \frac{\partial}{\partial x_{n-2p+2}} \}, \]
\[ x_n^{2p-3} \frac{\partial}{\partial x_{n-2p+1}}, Z \mid_{\mathbb{R}^{n-p-1}} \]

Similarly for \( E_{2p+1} \). Then we have (\( \beta \)).
Prove now \( \dim(E_{2(n-l)+1}) = \dim(E_{2(n-l)}) = \dim(E_{2(n-l)-1}) \), to obtain the 2nd step, because, by hypothesis, the small growth vector is \([2, 3, 4, \ldots, (n-1)_2, n]_S\) at 0. Discuss the two cases \( n - l \) is even or odd.

1) \( n-l \) is even: Let \( p = \frac{n-l-2}{2} \), by (\( \beta \))

\[ E_{2p+1} = E_{n-l-1} = \text{span}\{ \frac{\partial}{\partial x_n}, \ldots, \frac{\partial}{\partial x_{n+1/2}}, x_n \frac{\partial}{\partial x_{n+1/2}}, x_n^2 \frac{\partial}{\partial x_{n+1/2}}, x_n^3 \frac{\partial}{\partial x_{n+1/2}}, \ldots, x_n^{n-l-4} \frac{\partial}{\partial x_{n+1/2}}, Z \mid_{\mathbb{R}^{n+l-2}} \} \]

it gives
\[ E_{n-l} = \text{span}\{ \frac{\partial}{\partial x_n}, \ldots, \frac{\partial}{\partial x_{n+l/2}}, x_n \frac{\partial}{\partial x_{n+l/2}}, x_n^2 \frac{\partial}{\partial x_{n+l/2}}, x_n^3 \frac{\partial}{\partial x_{n+l/2}}, \ldots, x_n^{n-l-5} \frac{\partial}{\partial x_{n+l/2}}, Z \mid_{\mathbb{R}^{n+l-2}}, x_n^{n-l-3} X \} \]
\[ E_{n-l+1} = \text{span}\{ \frac{\partial}{\partial x_n}, \ldots, \frac{\partial}{\partial x_{n+l-1/2}}, x_n \frac{\partial}{\partial x_{n+l-1/2}}, x_n^2 \frac{\partial}{\partial x_{n+l-1/2}}, x_n^3 \frac{\partial}{\partial x_{n+l-1/2}}, \ldots, x_n^{n-l-6} \frac{\partial}{\partial x_{n+l-1/2}}, Z \mid_{\mathbb{R}^{n+l-2}}, x_n^{n-l-4} X, x_n^{n-l-2} [\frac{\partial}{\partial x_{n+l-1}}, X] \} \]
\[ E_{n-l+2} = \text{span}\{ \frac{\partial}{\partial x_n}, \ldots, \frac{\partial}{\partial x_{n+l-1/2}}, x_n \frac{\partial}{\partial x_{n+l-1/2}}, \ldots, x_n^{n-l-7} \frac{\partial}{\partial x_{n+l-1/2}}, Z \mid_{\mathbb{R}^{n+l-2}}, x_n^{n-l-5} X, x_n^{n-l-3} [\frac{\partial}{\partial x_{n+l-1}}, X], x_n^{n-l-1} x_{l+2} [X, [\frac{\partial}{\partial x_{n+l-1}}, X]] \} \]
similarly, we have for every \( i; 1 \leq i \leq \frac{n-l-6}{2} \)
We obtain,

\[ E_{n-l+2i} = \text{span} \left\{ \frac{\partial}{\partial x_n}, \ldots, \frac{\partial}{\partial x_{n-l+2i}}, x_n \frac{\partial}{\partial x_{n-l+2i}}, \ldots, x_n^{n-l-2i-5} \frac{\partial}{\partial x_{l+2}} \right\}, \]

\[ Z \mid x_{n-l+2i}, x_n^{n-l-2i-3}X, x_n^{n-l-2i-1} \left[ \frac{\partial}{\partial x_{l+1}}, X \right], x_n^{n-l-2i+1} x_{l+2} [X, \left[ \frac{\partial}{\partial x_{l+1}}, X \right]], x_n^{m} T_{n-l+2i}; m \geq n - l - 2i + 2 \}

\[ E_{n-l+2i+1} = \text{span} \left\{ \frac{\partial}{\partial x_n}, \ldots, \frac{\partial}{\partial x_{n-l+2i}}, \ldots, x_n^{n-l-2i-6} \frac{\partial}{\partial x_{l+2}}, Z \mid x_{n-l+2i+1}, \right\}, \]

\[ x_n^{n-l-2i-4} x_{l+2} [X, \left[ \frac{\partial}{\partial x_{l+1}}, X \right]], x_n^{n-l-2i+2} x_{l+2} [X, \left[ \frac{\partial}{\partial x_{l+1}}, X \right]], x_n^{m} T_{n-l+2i+1}; m \geq n - l - 2i + 1 \]

Remark that for \( t \geq 3 \), we add for \( E_{2t} \) the generators having the forms \( x_{n}^{m} T_{n-l+t}; m \geq n - l - t + 2 \). But \( x_{n}^{m} T_{n-l+t} \mid_{0} \) does not add another generator.

For \( i = \frac{n-l-6}{2} \), we have

\[ E_{2(n-l)-5} = \text{span} \left\{ \frac{\partial}{\partial x_n}, \ldots, \frac{\partial}{\partial x_{l+2}}, x_{l+2} X + \frac{\partial}{\partial x_{l+1}}, x_n^{2} X, x_n^{4} \left[ \frac{\partial}{\partial x_{l+1}}, X \right], \right\}, \]

\[ x_{n}^{6} x_{l+2} [X, \left[ \frac{\partial}{\partial x_{l+1}}, X \right]], x_{n}^{m} T_{2(n-l)-5}; m \geq 7 \]

Then,

\[ E_{2(n-l)-4} = \text{span} \left\{ \frac{\partial}{\partial x_n}, \ldots, \frac{\partial}{\partial x_{l+2}}, x_{l+2} X + \frac{\partial}{\partial x_{l+1}}, x_n X, x_n^{3} \left[ \frac{\partial}{\partial x_{l+1}}, X \right], \right\}, \]

\[ x_{n}^{5} x_{l+2} [X, \left[ \frac{\partial}{\partial x_{l+1}}, X \right]], x_{n}^{m} T_{2(n-l)-4}; m \geq 6 \}

\[ E_{2(n-l)-3} = \text{span} \left\{ \frac{\partial}{\partial x_n}, \ldots, \frac{\partial}{\partial x_{l+2}}, \frac{\partial}{\partial x_{l+1}}, X, x_n^{2} \left[ \frac{\partial}{\partial x_{l+1}}, X \right], \right\}, \]

\[ x_{n}^{4} x_{l+2} [X, \left[ \frac{\partial}{\partial x_{l+1}}, X \right]], x_{n}^{m} T_{2(n-l)-3}; m \geq 5 \}

We obtain,
\[ E_{2(n-l)-2} = \text{span}\{ \frac{\partial}{\partial x_n}, \ldots, \frac{\partial}{\partial x_{l+1}}, X, x_n[\frac{\partial}{\partial x_{l+1}}, X], x_n^3 x_{l+2}[X, [\frac{\partial}{\partial x_{l+1}}, X]], \\
x_n^m T_{2(n-l)-2}; m \geq 4 \} \]

\[ E_{2(n-l)-1} = \text{span}\{ \frac{\partial}{\partial x_n}, \ldots, \frac{\partial}{\partial x_{l+1}}, X, [\frac{\partial}{\partial x_{l+1}}, X], x_n^2 x_{l+2}[X, [\frac{\partial}{\partial x_{l+1}}, X]], \\
x_n^m T_{2(n-l)-1}; m \geq 3 \} \]

\[ E_{2(n-l)} = \text{span}\{ \frac{\partial}{\partial x_n}, \ldots, \frac{\partial}{\partial x_{l+1}}, X, x_n x_{l+2}[X, [\frac{\partial}{\partial x_{l+1}}, X]], \\
x_n^m T_{2(n-l)}; m \geq 2 \} \]

We deduce

\[ E_{2(n-l)+1} = \text{span}\{ \frac{\partial}{\partial x_n}, \ldots, \frac{\partial}{\partial x_{l+1}}, X, [\frac{\partial}{\partial x_{l+1}}, X], x_l+2[X, [\frac{\partial}{\partial x_{l+1}}, X]], \\
x_n^m T_{2(n-l)+1}; m \geq 1 \} \]

Then \( \dim(E_{2(n-l)+1}) = \dim(E_{2(n-l)}) = \dim(E_{2(n-l)-1}) \), impossible (because the small growth vector).

2) If \( n-l \) is odd: Let \( p = \frac{n-l-1}{2} \), in \( \beta \)

\[ E_{2p} = E_{n-l-1} = \text{span}\{ \frac{\partial}{\partial x_n}, \frac{\partial}{\partial x_{n+l+1}}, x_n \frac{\partial}{\partial x_{n+l-1}}, x_n^3 \frac{\partial}{\partial x_{n+3}}, \ldots, x_n^{n-l-4} \frac{\partial}{\partial x_{l+2}} \\
Z | \mathbb{R}^{n+l-1} \} \]

A similar proof to the even case shows that \( \dim(E_{2(n-l)+1}) = \dim(E_{2(n-l)}) = \dim(E_{2(n-l)-1}) \), impossible in reason of the small growth vector.

3rd step: 1) and 2) give

\[ E^+ = \begin{cases} \\
\omega_1 &= dx_2 + x_3 dx_1 \\
\omega_2 &= dx_3 + x_4 dx_1 \\
\omega_3 &= dx_4 + x_5 dx_1 \\
\omega_4 &= dx_5 + (C_6 + x_6) dx_1 \\
& \vdots \\
\omega_{n-3} &= dx_{n-2} + (C_n + x_{n-1}) dx_1 \\
\omega_{n-2} &= dx_1 + x_n dx_{n-1} \\
\end{cases} \]

and the theorem3 (in [2]) we transform the constants to 0.

**Remark:** To clarify the 2nd step, consider the following example:

Let \( E \) the 2-distribution spanned by

\[ v_1 = \frac{\partial}{\partial x_n}, \text{ and } v_2 = x_n Z + \frac{\partial}{\partial x_{n-1}}, \text{ with } Z = x_{n-1} X + \frac{\partial}{\partial x_{n-2}}. \]

In this case \( n-l=3 \) and \( \omega_{n-3}, \omega_{n-2} \) are 2 transpositions. We have:
\[ E_2 = \text{span}\{\frac{\partial}{\partial x^n}, \frac{\partial}{\partial x_{n-1}}, Z\} \]
\[ E_3 = \text{span}\{\frac{\partial}{\partial x^n}, \frac{\partial}{\partial x_{n-1}}, \frac{\partial}{\partial x_{n-2}}, X\} \]
\[ E_4 = \text{span}\{\frac{\partial}{\partial x^n}, \frac{\partial}{\partial x_{n-1}}, \frac{\partial}{\partial x_{n-2}}, X, x_n[\frac{\partial}{\partial x_{n-2}}, X]\} \]
\[ E_5 = \text{span}\{\frac{\partial}{\partial x^n}, \frac{\partial}{\partial x_{n-1}}, \frac{\partial}{\partial x_{n-2}}, X, [\frac{\partial}{\partial x_{n-2}}, X], x_n^2x_{n-1}[X, [\frac{\partial}{\partial x_{n-2}}, X]]\} \]
\[ E_6 = \text{span}\{\frac{\partial}{\partial x^n}, \frac{\partial}{\partial x_{n-1}}, \frac{\partial}{\partial x_{n-2}}, X, [\frac{\partial}{\partial x_{n-2}}, X], x_nx_{n-1}[X, [\frac{\partial}{\partial x_{n-2}}, X]], x_n^mT_6; m \geq 2\} \]
\[ E_7 = \text{span}\{\frac{\partial}{\partial x^n}, \frac{\partial}{\partial x_{n-1}}, \frac{\partial}{\partial x_{n-2}}, X, [\frac{\partial}{\partial x_{n-2}}, X], x_{n-1}[X, [\frac{\partial}{\partial x_{n-2}}, X]], x_n^mT_7; m \geq 1\} \]

Then \( \dim(E_{2(3)+1})_0 = \dim(E_{2(3)})_0 = \dim(E_{2(3)-1})_0, \) impossible.

References


Received: January, 2010