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Average Period of Production in Circulating Input-Output Structure

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Abstract

"Average period of production" is Böhm-Bawerk's concept to express the length of roundabout production processes. This concept has been criticized in that it can be applied only in the case of a single linear stage pattern of production. A method of how to determine average period of production in cases of circulating input-output, such as coal is necessary to produce steel and steel is necessary to produce coal, using the mathematics of the absorbing Markov chain, is proposed. Exploring other attemps at defining average period of production, a close relationship is found between this method, Marx's organic composition and the Frobenius root of the input coefficient matrix.

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Introduction[0]

Many concepts regarding technical intensiveness of capital have played important roles in economic theory. Mainstream textbooks predict that the relatively labor-rich countries will concentrate their production in labor-intensive sectors and relatively capital-rich countries will do so in capital-intensive sectors. Marx, as well as Ricardo, said that if real wage rates increase, prices of commodities of lower capital composition sectors will rise relative to those of commodities of higher capital composition sectors. Later, almost the same argument with this is espoused in the Stolper=Samuelson Theorem, which states

[0]I would like to thank Professor Shuhei Mitsuchi, Professor Seiji Nagata, Professor Christopher Bliss and anonymous referees of *Review of Austrian Economics* and *Metroeconomica* for their useful comments.

that higher prices of labor-intensive goods relative to prices of capital-intensive goods are accompanied by higher real wage rate and lower capital rental rate. In addition, there are many studies based on the concept of "capital coefficient" in the investment theory, the growth theory, etc.

But how should we measure these concepts of capital intensity? As is well known, it is difficult to calculate them from actual data.

First, if we aggregate capital goods by actual prices, the index of capital intensity fluctuates with the price, even if the technical intensiveness of capital remains constant. There may be cases in which a large increase of prices of some capital goods causes a slight labor-intensive technical change but shows a capital-intensive change in the calculated index.

And how can we compare the capital intensities of countries with different currencies? Is it not possible that technically higher capital-intensive countries would be calculated as lower capital-intensive ones by the effects of the exchange rate?

Second, as we will show later, we can indeed formulate a type of capital intensity concept independent of price fluctuations, according to Marx's notion of organic composition, which uses embodied labor values instead of the actual prices. However, as we will also show later, Marx's proposition of a price increase of lower organic composition goods with a real wage rate increase is false because of the insufficiency of the organic composition concept as a true capital intensity concept. For, the capital goods used in the higher organic composition sector may be produced by a very labor-intensive sector at the previous stage, whereas those used in the lower organic composition sector may be produced by a very capital-intensive sector at the previous stage. We cannot limit our focus only to the techniques of the final stage.

Third, how should we count the means of production of different durable periods? The capital intensity concept of mainstream economics counts only the fixed capital, but we should not omit counting the circulating capitals (intermediate goods). Continuing reproduction of intermediate goods of many stages is theoretically equivalent to durable fixed capital goods of staggered ages, because both need a certain amount of value to exist as physical forms, although both can recover newly reinvested money every year.

However, historically, a concept was once proposed regarding a type of intensiveness of capital clearing these difficulties. Is is known as "average period of production."

"Average period of production" is a concept used by Böhm-Bawerk to express the length of roundabout production processes. This notion has been criticized in that it can be applied only in the case of a single linear stage pattern of production as in Böhm-Bawerk's original examples, but not in more plausible cases of circulating input-output structure, such as coal is necessary to produce steel and steel is necessary to produce coal. However, the concept is a desirable index of capital intensity, because it is not directly affected by price fluctuations that disturb ordinary aggregation of capital goods. Moreover, it takes into account capital intensity of all stages of production not only that of the final stage. It would be useful for the modern economic theory of technical choice if we could somehow measure the "average period of production" of each final good in the real economy. In Matsuo (1994), I proposed a way of determinating the "average period of production" in the case of a circulating input-output structure, using the method of the Markov chain expansion. There Böhm-Bawerk's single linear structure case is shown as a special case. However, this work was written in Japanese. In this paper, this solution is introduced to non-Japanese readers, comparing it with similar concepts proposed by other authors.

Many interpretations of Böhm's capital theory, starting from Wicksell (1893), employ a macroeconomic production function which uses the period of production as input. However, such interpretations scarcely inquire as to the justification of extending this concept to general cases. Indeed, some authors have attempted to generalize Böhm's concept. Nonetheless, initial attemps by big names before the 1960s failed to succeed.

Hicks' (1946) "average period", which he states "the Austrians were looking for" (p.219), has nothing to do with Böhm-Bawerk's average period, although Hicks also wanted to extend this concept to general cases. Hicks' "average period" is that of gaining a profit from a certain investment, not of the production of a certain commodity as in Böhm-Bawerk's.

Knight (1935) and Kaldor (1937) defined the concept of average period of production as the ratio of the initial cost (amount of capital) to the annual maintenance cost. Dorfman (1959) and Blaug (1962) insisted that the capital coefficient of modern economics is the same concept; that is, the ratio of the amount of capital to the annual net output. These are interpreted by the physical logic of the "bathtub theorem,"[1] which states that the average time that water stays in a reservoir is given as a ratio, dividing the amount of water in the reservoir by the rate of flow into and out of the reservoir at unit time.

It seems that Knight (1935) and Kaldor (1937) shared the misunderstanding of Hicks (1946), confusing the period of production as the period of investment. Their concept of annual input is that of each investing firm, while Böhm's concept of annual input is the general equilibrium concept of the original input to produce each final product, which value is equal to the final output, as Dorfman (1959) and Blaug (1962) realized.

All four of these authors-Knight, Kaldor, Dorfman and Blaug-need to define the concept of amount of capital prior to determinate the average period concept. This is appropriate if we assume a fixed capital model that omits circulating capital. But Böhm's original model was that of circulating capital

[1] Blaug (1962, p.524).

without fixed capital. Thus, his concept of the amount of total capital invested cannot be predefined, especially for our circulating input-output structure.

After these attempts, Böhm's "period of production" was long forgotten. But in recent work, there appear some new attempts to re-establish this concept in modern economic techniques. The herald was Tintner (1974). He regarded the eigenvalue of the input coefficient matrix as a critical concept concerned with the period of production. However, this was defined only in the case of a triangular matrix. There was little explanation about the relationship with Böhm's original concept. Moreover, the argument was misconnected to the cyclical fluctuation period caused by the imaginary root of the eigenvalues. In spite of these defects, Tintner's proposal can be seen as a solution under certain conditions, as we shall see later.

Lager and Teixeira (2001) proposed to define the degree of "roundaboutness" as the ratio of the "value of total capital per unit value of direct capital employed," without interpretation of the relationship to Böhm's original concept. This also proved to be one solution under certain conditions, as we shall see later.

Besides Matsuo (1994), the most accurate solution so far has been proposed by Kurz and Salvadori (1995). Their formula is derived from a simple interest price equation, but to my regret, it is left in an intractable form to calculate actual value. If we develop the equation further, we can obtain an expression strictly equal to my Markov chain definition.

In Section 1, Böhm's concept of "average period of production" is described. In Section 2, this concept is expanded to the circulating input-output structure of the one sector corn model. In Section 3, how to determine the "average period of production" in the circulating input-output structure of a general n-sectors model, as proposed in Matsuo (1994), is demonstrated. In Section 4, this concept is shown to be consistent only with Böhm's approximation of simple interest and Kurz and Salvadori's concept is shown to be equal to mine. In Section 5, the relationship between this concept of period of production and the Marxian concept of "organic composition" is considered. In Section 6, under certain conditions, my concept of period of production and Lager and Teixeira's concept of degree of roundaboutness, as well as the Marxian concept of "organic composition" and the concept of "capital coefficient" of modern economics are shown to all arrive at an equivalent concept, the Frobenius root, which is one of Tintner's proposed eigenvalues. Equivalency between the propositions of Marx and Böhm on the diminishing interest (profit) rate is noted. Section 7 concludes.

1 BÖHM-BAWERK'S "AVERAGE PERIOD OF PRODUCTION" IN SINGLE LINEAR STRUCTURE

First, let us examine a simple example to explain Böhm's original concept of average period of production. Let us assume a final consumption good which requires three stages of production[2] . At each stage, one unit of labor is necessary to produce one unit of final good. The wage rate is $10. If one unit of final good is produced constantly each term, then at each term-end, one unit of first intermediate good, one unit of second intermediate good and one unit of final good are produced. So, under the assumption of wage advancement, the unrecovered money is $60. That is, $10 is embodied in the first intermediate good, $20 in the second intermediate good ($10 from the first intermediate good and $10 added at this stage), and $30 in the final good ($20 from the second intermediate good and $10 added at this stage). This $60 is regarded as the total invested capital.

In the total invested capital, $30 is embodied in the final good, and is recovered after it is sold. Exactly the same amount of money, $30, is invested to employ labor for the production of the next term, that is $10 for each stage. According to the physical logic of the "bathtub theorem" mentioned above, the average period of production must be given by dividing the total invested capital by the amount of money recovered and reinvested at each term-end. In this case, the average period of production is 2 (= $60/$30).

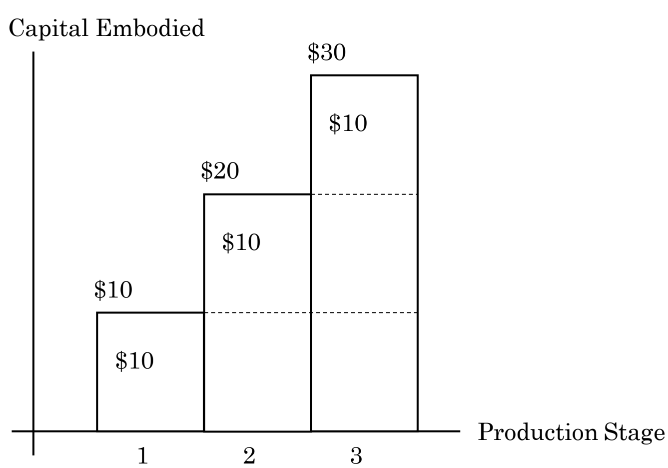

Figure 1: A Simple Example of Böhm's Original Concept

[2]Böhm-Bawerk's original example has 10 stages. See Böhm-Bawerk (1889, s. 95).

We can also explain this case as follows. See Fig. 1. From the \$30 of the invested money embodied in the final good at the end of the final stage, all \$30 is traced to one term back. We can say the "transition probability" of embodied money, to be traced back one term, is 1 at this stage. Within this amount, 2/3 of \$30 or \$20, is traced one term back. Here we can say the "transition probability" is 2/3 at this stage. From this money, 1/2 of \$20, or \$10, is traced one term back. The "transition probability" is 1/2 at this stage. Therefore, the expected value of total terms traced back can be given as $1 + 2/3 + (2/3) \bullet (1/2) = 2$. This is the average period of production.

Instead of money expression, we can also express the "transition probability" as ratio of embodied labor of previous stage to the total labor embodied at the stage, because the wage rate can be canceled out as both numerator and denominator.

2 "AVERAGE PERIOD OF PRODUCTION" IN ONE-SECTOR CIRCULATING INPUT-OUTPUT STRUCTURE[3]

Let us suppose a special product called "corn". To produce one unit of "corn", we must input a units of "corn" and l units of direct labor. "Corn" is necessary to produce "corn". So this is the simplest case of the circulating input-output structure. In this case, Fig. 1 can be redrawn as Fig. 2.

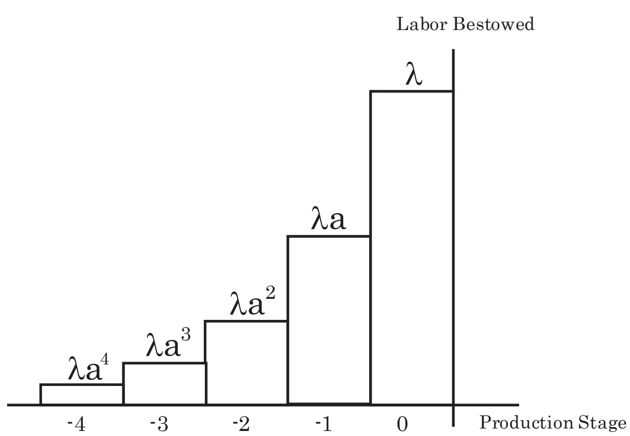

Figure 2: Case of "Corn"

λ is the amount of direct and indirect labor bestowed in one unit of "corn". The final stage is stage 0. Here a new l unit of labor is added, and λa,

[3]First I presented the argument of this section in Matsuo (1994).

the amount of labor bestowed on a units of "corn", has been passed from stage -1. At this stage, al units of labor was added, and λa^2 , the amount of labor bestowed in a^2 units of "corn", has been passed from the stage -2. This continues backward infinitely.

Therefore, in λ units of labor bestowed in one unit of final product, all λ is traced back one term. The "transition probability" from the end of stage 0 to the end of stage -1 is one. And in the λ units of labor, λa is traced back one term. The "transition probability" from the end of stage -1 to the end of stage -2 is $a = \lambda a/\lambda$. The "transition probability" from the end of stage 0 to the end of stage -2 is $a = a \times 1$. Under the same consideration, for $t > 2$, the "transition probability" from the end of stage $-t$ to the end of stage $-(t+1)$ is always a and the "transition probability" from the end of stage 0 to the end of stage $-(t+1)$ is a^t .

Thus, the expected value of total terms, traced back from the bestowed labor in the final product can be given as

$$1 + a + a^2 + a^3 + \cdots = \frac{1}{1-a} \tag{1}$$

This is the average period of production. Of course, $a < 1$ from the possibility of reproduction guarantees convergence. Therefore, the average period of production is obtained as a finite value, though we must trace back the production stages infinitely.

3 "AVERAGE PERIOD OF PRODUCTION" IN GENERAL CIRCULATING INPUT-OUTPUT STRUCTURE

In a general circulating structure of production, the proportion of labor traced back to the i-th sector, sharing in the total labor embodied in the j-th sector product, can be written as $\frac{\lambda_i a_{ij}}{\lambda_j}$. This is the "transition probability" to the i-th sector at the j-th sector.

Here, λ_i is direct indirect embodied labor in one unit of commodity of i-th sector and defined here as $\boldsymbol{\lambda} = \boldsymbol{\lambda}\mathbf{A} + \boldsymbol{l}$, where $\boldsymbol{\lambda} = (\lambda_1, \lambda_2, \cdots, \lambda_n)$, $\boldsymbol{l} = (l_1, \cdots, l_n)$ and $\mathbf{A}$ is the input coefficient matrix, the i-th row, j-th column element of which is a_{ij}.

Therefore, we can compose a "matrix of transition probabilities" which has this "transition probability" as its element of the i-th row, j-th column. Denoting this matrix as $\mathbf{Q}$, it is defined as

$$\mathbf{Q} = \boldsymbol{\Lambda}\mathbf{A}\boldsymbol{\Lambda}^{-1} \tag{2}$$

where, $\mathbf{\Lambda}$ is a diagonal matrix, the i-th element of which is λ_i. The element of the i-th row, j-th column of $\mathbf{Q}^n$ is the probability of being at the i-th sector traced n terms back from total labor bestowed in the final j-th goods. Therefore, the sum of each column of $\mathbf{Q}^n$, which is expressed as $\mathbf{eQ}^n$, where $\mathbf{e}$ is the unit vector (all elements are unity), means the probability of being at any sector traced n terms back from the total labor bestowed in the final j-th goods.

Then, as we can easily infer from the argument of the previous section, from the mathematics of an absorbing Markov chain (see Bradley and Meek (1986)), the average period of production can be obtained as follows[4],

$$\boldsymbol{\theta} = \mathbf{e}(\mathbf{I} + \mathbf{Q} + \mathbf{Q}^2 + \mathbf{Q}^3 + \cdots) \tag{3}$$
$$= \mathbf{e}(\mathbf{I} - \mathbf{Q})^{-1} \tag{4}$$

where $\boldsymbol{\theta} = (\theta_1, \theta_2, \cdots, \theta_n)$ is the vector, elements of which are the average period of production of each commodity. Or inserting (2) into (4),

$$\boldsymbol{\theta} = \mathbf{e} + \mathbf{eQ} + \mathbf{eQ}^2 + \mathbf{eQ}^3 + \cdots \tag{5}$$
$$= \mathbf{e} + \boldsymbol{\lambda}\mathbf{A}\mathbf{\Lambda}^{-1} + \boldsymbol{\lambda}\mathbf{A}\mathbf{\Lambda}^{-1}\mathbf{\Lambda}\mathbf{A}\mathbf{\Lambda}^{-1} + \cdots \tag{6}$$
$$= \mathbf{e} + \boldsymbol{\lambda}\mathbf{A}\mathbf{\Lambda}^{-1} + \boldsymbol{\lambda}\mathbf{A}^2\mathbf{\Lambda}^{-1} + \boldsymbol{\lambda}\mathbf{A}^3\mathbf{\Lambda}^{-1} + \cdots \tag{7}$$
$$= \boldsymbol{\lambda}(\mathbf{I} - \mathbf{A})^{-1}\mathbf{\Lambda}^{-1} \tag{8}$$

Convergence is guaranteed because $\mathbf{A}$ satisfies the Hawkins-Simon's condition of the possibility of reproduction.

Thus, we can extend Böhm's concept of average period of production to a circulating input-output structure. If we suppose such an input coefficient matrix as $a_{i-1\,i} > 0$ for $i = 2$ to n, and all other elements are equal to zero, then we can obtain Böhm-Bawerk's original concept of single linear system from (8).

4 "AVERAGE PERIOD OF PRODUCTION" AND SIMPLE INTEREST APPROXIMATION

As Kaldor (1937) and Blaug (1962) pointed out[5] , this determination of average period is consistent only with simple interest approximation, which Böhm-Bawerk supposes throughout his arguments. The example in Section 1 is the same as the situation in which one always has bonds of one-year maturity, two-years maturity and three-years maturity of $10 each. If one invests this

[4]This expression is what I first presented in Matsuo (1994). The expression of (8) is first presented here.

[5]Blaug (1962, pp. 519-52). See also Kurz and Salvadori (1995, p. 437).

$30 into a single type of bond which has the same advantage as this situation, how many years until maturity does the bond have? The answer is the solution of θ_* in the equation below.

$$(1+r)^{\theta_*}30 = (1+r)10 + (1+r)^2 10 + (1+r)^3 10 \tag{9}$$

where r is the interest rate. This is the true average period of production. As Blaug (1962) states, this value depends on the interest rate[6]. If we take a simple interest approximation of the above equation, we obtain the equation below.

$$(1+\theta r)30 = (1+r)10 + (1+2r)10 + (1+3r)10 \tag{10}$$

Solving this equation, we obtain $\theta = 2$. This is Böhm's average period of production.

We can show that the generalized concept of average period of production demonstrated in Section 3 is also consistent only with Böhm's approximation of simple interest.

First, we shall define the function as follows; let $\alpha \in (-1, 1)$ be scalar and $\mathbf{X}$ be a square matrix, define the function of scalar to a power of matrix as,

$$(1+\alpha)^{\mathbf{X}} \equiv 1 + \alpha\mathbf{X} + \frac{1}{2!}\alpha^2\mathbf{X}(\mathbf{X}-\mathbf{I}) + \frac{1}{3!}\alpha^2\mathbf{X}(\mathbf{X}-\mathbf{I})(\mathbf{X}-2\mathbf{I}) + \cdots \tag{11}$$

From this definition, if $\mathbf{X}$ is a diagonal matrix and the i-th element χ_i, then $(1+\alpha)^{\mathbf{X}}$ is the diagonal and the i-th element is $(1+\alpha)^{\chi_i}$.

We can then define the "true" average period of production θ_i^* as follows.

$$\boldsymbol{p} = w\boldsymbol{\lambda}(1+r)^{\boldsymbol{\Theta}^*} \tag{12}$$

where $\boldsymbol{p}$ is the "production price[7]" vector, defined as $\boldsymbol{p} = (1+r)(\boldsymbol{p}\mathbf{A} + w\boldsymbol{l})$, and $\boldsymbol{\Theta}^*$ is the diagonal matrix, the i-th element of which is θ_i^* .

We cannot solve this equation analytically to obtain θ_i^*. We can, however, use θ_i instead, under a small interest rate. Because the following proposition now holds.

> *Prop.4-1* For any commodity i, the average period of production θ_i is the simple interest approximation of the "true" average period of production θ_i^*.

[6]Kaldor (1937) also realized this in his interpretation of "period of investment". See Kaldor (1937, p. 213, Footnote 21).

[7]"Production price" is the Marxian term for the price under uniform profit rate. This concept is identical to Böhm's price concept, because his concept of interest rate is the same as the uniform profit rate of Marx.

(Proof) From the definition of the production price vector, we can get,[8]

$$\boldsymbol{p} = (1+r)w\boldsymbol{l}[\mathbf{I}-(1+r)\mathbf{A}]^{-1} \quad (13)$$
$$= (1+r)w\boldsymbol{l}+(1+r)^2 w\boldsymbol{l}\mathbf{A}+(1+r)^3 w\boldsymbol{l}\mathbf{A}^2+\cdots \quad (14)$$

From (11)

$$w\boldsymbol{\lambda}(1+r)^{\boldsymbol{\Theta}^*} = w\boldsymbol{\lambda}\left[\mathbf{I}+\boldsymbol{\Theta}^* r+\frac{1}{2!}\boldsymbol{\Theta}^*(\boldsymbol{\Theta}^*-\mathbf{I})r^2+\cdots\right] \quad (15)$$

Taking a linear approximation around the $r=0$ point, inserting it into (12), and denoting this approximated value by removing "*", we obtain (8). For details, see Appendix. (q.e.d.)

[Prop. 4-1] was first proposed by Kurz and Salvadori (1995)[9] not as a proposition but as the definition of "average period of production." From a demonstration identical to that mentioned above, they obtained the "average period of production" of the i-th commodity as,

$$\theta_i = \frac{\sum_{n=1}^{\infty} n l_{n\,i}}{\lambda_i} \quad (16)$$

where, $l_{n\,i}$ is the i-th element of $\mathbf{A}^n\boldsymbol{l}$. This concept is equivalent to my concept but it is an expression obtained directly from the first line found in the Appendix. In this form, it needs infinite summation to calculate the actual values, whereas our (8) provides the values to us at once.

The "true" average period of production depends directly on interest rate, while our approximated one does not. This relationship with interest rate causes the famous re-switching example (Samuelson, 1966) of the Cambridge capital controversy.

5 "AVERAGE PERIOD OF PRODUCTION" AND MARX'S ECONOMIC CONCEPT

The concept of so-called capital intensity is difficult to measure when there are more than two types of means of production. If we aggregate these means of production by their prices, capital intensity will fluctuate with factor price fluctuation even when the technique is constant. Böhm-Bawerk considered his

[8] This expansion is the same as that of Sraffa's (1960) "reduction to dated quantities of labour". He recognized that this expansion is related to the concept of average period of production. However, he did not recognize its significance for the capital theory and did not examine it further (*ibid.* p. 38).

[9] pp. 437.

idea of average period of production as an index of capital intensity, independent of price fluctuation because he thought a longer roundabout production process meant higher equipment per capita[10].

Another concept of an index of capital intensity independent of price fluctuation is "organic composition", originally employed by Marx under the name of "organic composition of capital" and revised by Okishio under the name of "organic composition of production." The former is defined as the "value of means of production/value of labor power." However, this is dependent on distribution. Marx wanted to limit this concept to that which reflects technical composition. So Okishio proposed the latter concept, defined as "labor embodied in means of production/labor added directly" or "dead labor/living labor."[11]Let us examine here the relationship between the average period of production and the organic composition of production.

The organic composition of production of the j-th sector is written as

$$\frac{\sum_{i=1}^{n} \lambda_i a_{ij}}{l_j}$$

Here we shall use an equivalent concept to this, defined as γ_j below.

$$\gamma_j = \frac{\sum_{i=1}^{n} \lambda_i a_{ij}}{\lambda_j} = \frac{\sum_{i=1}^{n} \lambda_i a_{ij}}{\sum_{i=1}^{n} \lambda_i a_{ij} + l_j}$$

We shall call this "organic composition rate," which is "dead labor/whole labor embodied." The organic composition rate is a one-to-one increasing function of the organic composition of production. From the definition of $\mathbf{Q}$, row vector $\boldsymbol{\gamma}$, the i-th element of which is γ_i, can be written as

$$\boldsymbol{\gamma} = \mathbf{eQ} \tag{17}$$

As we saw in (4), $\boldsymbol{\theta} = \mathbf{e} + \mathbf{eQ} + \mathbf{eQ}^2 + \mathbf{eQ}^3 + \cdots$. Thus, the organic composition rate of each sector is the first-order approximation of the Markov chain expansion of the average period of production of each commodity. Using the concept of organic composition, Marx proposed these three propositions in his *Capital*, Book 3, as indicated below.

[Existing Prop.1] $\boldsymbol{p} = \mu\boldsymbol{\lambda} \Leftrightarrow \boldsymbol{\gamma} = \gamma\mathbf{e}$
where, μ and γ are adequate positive scalar. That is, production prices are

[10] Mitani (1942, pp. 153-154).
[11] Okishio (1993, p. 361)

proportional to labor values if and only if the organic composition of each sector is uniform.

[False Prop. 1] $\frac{p_i}{p_j} \gtreqless \frac{\lambda_i}{\lambda_j} \Leftrightarrow \gamma_i \gtreqless \gamma_j$
That is, the relative price of the commodity of a higher organic composition sector to that of a lower organic composition sector is higher than the ratio of the labor values of both commodities[12].

[False Prop. 2] $\frac{d(p_i/p_j)}{dr} \gtreqless 0 \Leftrightarrow \gamma_i \gtreqless \gamma_j$
That is, if the real wage rate increases[13], the relative price of the commodity of a lower organic composition sector increases. (As known, profit rate[14], r, has a negative relationship with the real wage rate.)

Both of these two propositions above have counter examples[15]. But if we substitute the concept of average period of production for the concept of organic composition, these three examples hold, i.e.,

Prop.5-1 $\boldsymbol{\gamma} = \gamma\mathbf{e} \Leftrightarrow \boldsymbol{\theta} = \theta\mathbf{e}$
where γ and θ are adequate positive scalar.

That is, the uniformity of the organic composition is identical to the uniformity of the average period of production.

(Proof)
i) Proof of $\boldsymbol{\gamma} = \gamma\mathbf{e} \Leftrightarrow \boldsymbol{\theta} = \theta\mathbf{e}$
From (6-1),

$$\begin{aligned}
&\mathbf{eQ} = \gamma\mathbf{e} \\
&\mathbf{eQ}^2 = \gamma\mathbf{eQ} = \gamma^2\mathbf{e} \\
&\mathbf{eQ}^3 = \gamma^2\mathbf{eQ} = \gamma^3\mathbf{e} \\
&\vdots
\end{aligned}$$

$$\begin{aligned}
\therefore \boldsymbol{\theta} &= \mathbf{e} + \mathbf{eQ} + \mathbf{eQ}^2 + \mathbf{eQ}^3 + \cdots \\
&= \mathbf{e} + \gamma\mathbf{e} + \gamma^2\mathbf{e} + \gamma^3\mathbf{e} + \cdots \\
&= (1 + \gamma + \gamma^2 + \gamma^3 + \cdots)\mathbf{e}
\end{aligned}$$

[12]Marx (1964, s. 174).
[13]Marx (1964, s. 210-212).
[14]This is Marx's "average profit rate", which is the same as Böhm's interest rate (hereafter called "uniform profit rate").
[15]Nakatani (1994, pp. 56-58).

from the definition of $\gamma_i, \gamma \in (0,1)$. Thus,

$$\boldsymbol{\theta} = \frac{1}{1-\gamma}\mathbf{e}$$

(q.e.d.)

ii) Proof of$\boldsymbol{\gamma} = \gamma\mathbf{e} \Leftarrow \boldsymbol{\theta} = \theta\mathbf{e}$

$$\boldsymbol{\theta}\mathbf{Q} = \theta\mathbf{e}\mathbf{Q}$$

From the definition,$\boldsymbol{\theta} = \boldsymbol{\theta}\mathbf{Q} + \mathbf{e}$. Thus,

$$\theta\mathbf{e} = \theta\mathbf{e}\mathbf{Q} + \mathbf{e}$$

From the definition,$\theta_i > 1$.

$$\therefore \boldsymbol{\gamma} = \mathbf{e}\mathbf{Q} = \frac{\theta - 1}{\theta}\mathbf{e}$$

(q.e.d.)

Prop.5-2 $\boldsymbol{p} = \mu\boldsymbol{\lambda} \Leftrightarrow \boldsymbol{\theta} = \theta\mathbf{e}$
where μ and θ are adequate positive scalar.

That is, production prices are proportional to labor values if and only if the average period of production of each commodity is uniform.

(Proof) From [Existing Prop. 1] and [Prop. 5-1].

(q.e.d.)

Prop.5-3 Under simple interest approximation,

$$\frac{p_i}{p_j} \gtreqless \frac{\lambda_i}{\lambda_j} \Leftrightarrow \theta_i \gtreqless \theta_j$$

That is, under simple interest approximation, the relative price of a commodity of the longer average period of production to that of the shorter one is higher than the ratio of the labor values of both commodities.

(Proof) From [Prop. 4-1], $p_i = (1 + r\theta_i)w\lambda_i$under simple interest. Therefore,

$$\frac{p_i}{p_j} = \frac{(1 + r\theta_i)\lambda_i}{(1 + r\theta_j)\lambda_j}$$

(q.e.d.)

Prop.5-4 Under simple interest approximation,

$$\frac{d(p_i/p_j)}{dr} \gtreqqless 0 \Leftrightarrow \theta_i \gtreqqless \theta_j$$

That is, if the real wage rate increases, the relative price of the commodity of the shorter average period of production increases.

(Proof)

$$\begin{aligned}\frac{d\log(p_i/p_j)}{d\log r} &= \frac{\theta_i\lambda_i}{\lambda_i + r\theta_i\lambda_i} - \frac{\theta_j\lambda_j}{\lambda_j + r\theta_j\lambda_j} \\ &= \frac{1}{\frac{1}{\theta_i} + r} - \frac{1}{\frac{1}{\theta_j} + r} \gtreqqless 0 \Leftrightarrow \theta_i \gtreqqless \theta_j\end{aligned}$$

(q.e.d.)

If we use the "true" average period of production defined by (12), [Prop. 5-3] and [Prop. 5-4] hold without simple interest approximation. So we can conclude that Böhm-Bawerk's concept of average period of production is a strict rendering of Marx's idea of organic composition.

The argument above concerns each sector. From a macroeconomic viewpoint, the concept of average period of production is identical to the concept of organic composition of production under a certain type of aggregation. That is to say,

Prop.5-5 Using the ratio of labor embodied in each final good to the total labor bestowed as the weight of summation, the average "average period of production" of total final goods is identical to the organic composition of production of the whole economy plus 1.

(Proof) Let average "average period of production" of total final goods be $\bar{\theta}$, gross outputs vector be $\mathbf{x}$ and net outputs vector be $\mathbf{y} = (\mathbf{I} - \mathbf{A})\mathbf{x}$, then,

$$\bar{\theta} = \frac{\sum_{i=1}^{n} \theta_i\lambda_i y_i}{\boldsymbol{l}\mathbf{x}} = \frac{\boldsymbol{\theta}\boldsymbol{\Lambda}\mathbf{y}}{\boldsymbol{l}\mathbf{x}} \tag{18}$$

Inserting (8),

$$= \frac{\boldsymbol{\lambda}(\mathbf{I} - \mathbf{A})^{-1}\boldsymbol{\Lambda}^{-1}\boldsymbol{\Lambda}\mathbf{y}}{\boldsymbol{l}\mathbf{x}} = \frac{\boldsymbol{\lambda}\mathbf{x}}{\boldsymbol{l}\mathbf{x}} \tag{19}$$

$$= \frac{\boldsymbol{\lambda}\mathbf{A}\mathbf{x}}{\boldsymbol{l}\mathbf{x}} + 1 \tag{20}$$

$\boldsymbol{\lambda}\mathbf{A}\mathbf{x}$ is total "dead labor" used in the whole economy and $\boldsymbol{l}\mathbf{x}$ is total "living labor" bestowed in the whole economy. So the first term of the right-hand side of the equation is Okishio's concept of organic composition of production of the whole economy.

(q.e.d.)

Böhm-Bawerk's concept of total invested capital of the whole economy is $w\boldsymbol{\lambda}\mathbf{x}$, because not only $w\boldsymbol{\lambda}\mathbf{A}\mathbf{x}$ but also wages paid in advance for the workers at the final stage must be included in total invested capital. Therefore, $\bar{\theta}$ means total invested capital divided by the amount of money recovered and reinvested at each term-end in the whole economy. This is consistent with the "bathtub theorem". As mentioned above, Dorfman (1959) and Blaug (1962) believed from the analogy of the "bathtub theorem" that the capital coefficient of modern economics and the average period of production were the same concept[16], while Okishio shows his concept of organic composition of production is approximately equal to that of the capital coefficient[17]. So we can say [Prop. 5-5] is a justification of what Dorfman (1959) and Blaug (1962) wanted to say.

6 "AVERAGE PERIOD OF PRODUCTION" AND THE FROBENIUS ROOT

Our definition of the average period (8) is a loyal extension of Böhm-Bawerk's original concept. But apart from the concept of Böhm-Bawerk, we can compose similar "average period" concepts using any value vector instead of the direct-indirect bestowed labor vector in (8).

Here, let us define a new concept of "average period of production". Let $\bar{\mathbf{p}}$ be the left Frobenius vector of $\mathbf{A}$. Use $\bar{\mathbf{p}}$ instead of $\boldsymbol{\lambda}$ in (8). Then we can compose a new "average period" vector $\boldsymbol{\theta}^{\mu}$ as follows.

$$\boldsymbol{\theta}^{\mu} = \bar{\mathbf{p}}(\mathbf{I} - \mathbf{A})^{-1}\bar{\mathbf{P}}^{-1} \tag{21}$$

Here, $\bar{\mathbf{P}}$ is the diagonal matrix, elements of which are those of $\bar{\mathbf{p}}$. This new concept also satisfies Böhm's search for an index of capital intensity, independent of price fluctuation.

Then using $\boldsymbol{\theta}^{\mu}$, we can also define the new average "average period of production" of total final goods $\bar{\theta^{\mu}}$, in accordance with (20). Here the weight

[16] Blaug (1962, p. 525).
[17] Okishio (1993, p. 377).

of summation must be the ratio of the value of each final good to that of the total final goods measured by $\bar{\mathbf{p}}$.

$$\bar{\theta^\mu} = \frac{\bar{\mathbf{p}}(\mathbf{I}-\mathbf{A})^{-1}\bar{\mathbf{P}}^{-1}\bar{\mathbf{P}}\mathbf{y}}{\bar{\mathbf{p}}\mathbf{y}} = \frac{\bar{\mathbf{p}}\mathbf{x}}{\bar{\mathbf{p}}\mathbf{y}} \tag{22}$$

$$= \frac{\bar{\mathbf{p}}\mathbf{x}}{\bar{\mathbf{p}}(\mathbf{I}-\mathbf{A})\mathbf{x}} = \frac{\bar{\mathbf{p}}\mathbf{x}}{\bar{\mathbf{p}}\mathbf{x}-\bar{\mathbf{p}}\mathbf{A}\mathbf{x}} = \frac{\bar{\mathbf{p}}\mathbf{x}}{\bar{\mathbf{p}}\mathbf{x}-\mu\bar{\mathbf{p}}\mathbf{x}} \tag{23}$$

$$= \frac{1}{1-\mu} \tag{24}$$

Here μ is the Frobenius root of $\mathbf{A}$. Therefore, $\bar{\theta^\mu}$ is a one-to-one increasing function of the Frobenius root of the input coefficient matrix.

As mentioned in the introduction of this paper, Tintner (1974) regards eigenvalues of $\mathbf{A}$ as a reflection of the period of production. Now we know that this is in a sense true for the maximum eigenvalue.

As (22) holds for any $\mathbf{y}$, it holds for $\mathbf{y} = (0, 0, \cdots, 0, 1, 0, \cdots, 0)$, where the i-th element is unity and others are zero, for any i. Thus,

$$\boldsymbol{\theta}^\mu = (\bar{\theta^\mu}, \bar{\theta^\mu}, \cdots, \bar{\theta^\mu}) \tag{25}$$

That is, the "average period of production" in this sense is uniform for any commodity[18] .

On the other hand, if we use the right Frobenius vector of $\mathbf{A}$ as the quantity vector to calculate the average "average period of production" of total final goods, then not only under $\bar{\mathbf{p}}$, but also under any positive value vector, the average "average period" of total final goods will be $1/(1-\mu)$. Of course, $\bar{\theta}$ of (20) also becomes $1/(1-\mu)$.

Lager and Teixeira (2001) provide their "degree of roundaboutness" as $\frac{\mathbf{pHx}}{\mathbf{pAx}}$, where $\mathbf{H} \equiv (\mathbf{I} - \mathbf{A}^{-1})\mathbf{A}$ is called the "matrix of vertical integrated capital inputs", which shows that if $\mathbf{x}$ is the right Frobenius vector of $\mathbf{A}$, then their "degree of roundaboutness" also becomes $1/(1-\mu)$. This is understandable, because if we substitute $\mathbf{y}$ for $\mathbf{Ax}$, then this becomes the same form as (18) or (22), and if $\mathbf{x}$ is the Frobenius vector, then $\mathbf{x}$, $\mathbf{y}$ and $\mathbf{Ax}$ are not different except in scale[19].

[18] Sraffa (1960) states that under the maximum uniform profit rate, "the value-ratios of net product to means of production" of all industries are equal to the same maximum profit rate (*ibid.* p.17). This situation is equivalent as we see here. Sraffa applied this "value ratio", the inverse of which is the capital coefficient of the "bathtub" interpretation, as a "proportion" index of capital intensity. An alternative ratio he provides is a ratio of "direct to indirect labour employed", the inverse of which is Okishio's organic composition of production (*ibid.* p. 16).

[19] If $\mathbf{x}$ is the Frobenius vector, it indicates the proportions of gross outputs, which grow with the maximal rate.

If we apply the actual price vector instead of $\bar{\mathbf{p}}$ of (22), the average "average period of production" of total final goods is the capital coefficient plus 1. Thus, in this case, the interpretation by Dorfman (1959) and Blaug (1962) that equates the average period of production to the capital coefficient has a strict basis. And if we apply the right Frobenius vector of $\mathbf{A}$ for the quantity vector to calculate the average "average period of production" of total final goods, then this value, Marx-Okishio's "organic composition" (plus 1) and the capital coefficient of modern economics (plus 1) as well as Lager and Teixeira's (2001) "degree of roundaboutness" all become identical to $1/(1-\mu)$.

As is well known, $\mu = 1/(1+r_M)$, where r_M is the maximum profit rate. Therefore, $\bar{\theta^{\mu}}$ under general quantity vectors or other values of average "average period of production" of the Frobenius quantity vector of $\mathbf{A}$, are equal to $1/r_M + 1$. Thus the maximum profit rate falls when the value of average period of production increases.

Marx stated that the general profit rate falls as the organic composition increases. Many authors have criticized him saying that this movement could be cancelled out by increasing the capital distribution rate. Against that criticism, Okishio defended Marx's logic saying that the organic composition of production is the inverse of the maximum profit rate, which must decrease as the organic composition increases. Thus, if the maximum profit rate falls, the general profit rate must fall in the long run[20].

Strictly speaking, the equivalency between the organic composition and the inverse of the maximum profit rate is an approximation. Under the production of the Frobenius quantity vector of $\mathbf{A}$, Okishio's argument strictly holds. On the other hand, Böhm-Bawerk considered that the interest rate falls as the average period of production increases. From [Prop. 5-5], this is equivalent to that of Marx's falling profit, as a macroeconomic argument. Therefore, it also holds, only by Okishio's falling maximum profit rate logic, and strictly requires the condition of the production of the Frobenius quantity vector of $\mathbf{A}$ or the average "average period of production" measured by the left Frobenius value vector.

7 CONCLUDING REMARKS

Thus Böhm-Bawerk's concept of "average period of production" has been successfully extended to the general case of circulating input-output structure of production. We have confirmed that Marx's arguments on capital composition and prices hold true by using a more appropriate concept of average period of production. We have also confirmed the tight relationship between the average period of production, Marx-Okishio's organic composition of production

[20] Okishio (1993) pp. 378-379.

and the Frobenius root of the input coefficient matrix. From this, we can observe equivalency between Marx's falling profit rate proposition and Böhm's falling interest rate proposition. It is ironical that the concept of the most severe criticizer against Marx, which had long been forgotten by contemporary economists, has been revived to support the Marxian concept.

The research of this paper is only the first step. In order to make use of this concept for the development of economic theories concerning capital intensiveness, we must measure the average period of production of each product from actual data.

The problems that are yet to be considered for this purpose are as follows:
(1) Extending the notion to cases of fixed capital or general joint production.
(2) Extending the notion to cases of heterogeneous labor.

For (1), I think we can solve the fixed capital problem using a dummy sector model. The simplest way is to build a "fixed capital sector" at the I-O matrix, considering the investment column as inputs from other sectors, depreciation row as inputs to other sectors, and undepreciated parts of the whole fixed capital as inputs to itself. More detailed analysis can be achieved by disintegrating this dummy good to particular fixed capital goods. Moreover, I think (2) can be solved by the model, in which complex labors are produced in the system (e.g. by the products of the education sector, etc.).

Lastly, instead of labor, we can determine the similar concept of "average period of production" by any commodity as the original input, for example, water or crude oil. Although this is not the case analyzed by ordinary economics, including that by Böhm-Bawerk, it may be useful depending on some particular problems.

APPENDIX

Taking the linear approximation of (12) and (14) at around $r = 0$, inserting them into (15) and denoting this approximated value as removing "*", we obtain **I**

$$
\begin{aligned}
w\boldsymbol{\lambda}(\mathbf{I}+\boldsymbol{\Theta}t) &= (1+r)w\boldsymbol{l} + (1+2r)w\boldsymbol{l}\mathbf{A} + (1+3r)w\boldsymbol{l}\mathbf{A}^2 + \cdots \\
&= (w\boldsymbol{l} + w\boldsymbol{l}\mathbf{A} + w\boldsymbol{l}\mathbf{A}^2 + \cdots) \\
&\quad +\{rw\boldsymbol{l} + rw\boldsymbol{l}\mathbf{A} + rw\boldsymbol{l}\mathbf{A}^2 + \cdots\} \\
&\quad\quad +[rw\boldsymbol{l}\mathbf{A} + rw\boldsymbol{l}\mathbf{A}^2 + \cdots] \\
&\quad\quad\quad +\langle rw\boldsymbol{l}\mathbf{A}^2 + \cdots\rangle \\
&= (w\boldsymbol{l}(\mathbf{I}-\mathbf{A})^{-1}) + \{rw\boldsymbol{l}(\mathbf{I}-\mathbf{A})^{-1}\} \\
&\quad +[rw\boldsymbol{l}(\mathbf{I}-\mathbf{A})^{-1}\mathbf{A}] + \langle rw\boldsymbol{l}(\mathbf{I}-\mathbf{A})^{-1}\mathbf{A}^2\rangle + \cdots
\end{aligned}
$$

From the definition of $\boldsymbol{\lambda}$, $\boldsymbol{\lambda} = \boldsymbol{l}(\mathbf{I}-\mathbf{A})^{-1}$. Thus,

$$
\begin{aligned}
w\boldsymbol{\lambda}(\mathbf{I}+\boldsymbol{\Theta}r) &= w\boldsymbol{\lambda} + rw\boldsymbol{\lambda} + rw\boldsymbol{\lambda}\mathbf{A} + rw\boldsymbol{\lambda}\mathbf{A}^2 + \cdots \\
&= w\boldsymbol{\lambda} + rw\boldsymbol{\lambda}(\mathbf{I}-\mathbf{A})^{-1} \\
rw\boldsymbol{\lambda}\boldsymbol{\Theta} &= rw\boldsymbol{\theta}\boldsymbol{\Lambda} = rw\boldsymbol{\lambda}(\mathbf{I}-\mathbf{A})^{-1} \\
&\therefore \boldsymbol{\theta} = \boldsymbol{\lambda}(\mathbf{I}-\mathbf{A})^{-1}\boldsymbol{\Lambda}^{-1}
\end{aligned}
$$

This is same as (8). (q.e.d.)

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