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On the Degree Sequence of Random Geometric Digraphs

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Abstract

A random geometric digraph G_n is constructed by taking $\{X_1, X_2, \cdots, X_n\}$ in $\mathbb{R}^2$ at random with a common bounded density function. Each vertex X_i is assigned at random a sector S_i of central angle α with inclination Y_i, in a circle of radius r (with vertex X_i as the origin). An arc is present from vertex X_i to X_j, if X_j falls in S_i. Suppose k is fixed and $\{k_n\}$ is a sequence with $1 \ll k_n \ll n^{1/2}$, as $n \to \infty$. We prove degree distributions as well as central limit theorems for $k-$ and k_n-nearest neighbor distance of out/in-degrees in G_n.

Mathematics Subject Classification: 05C80

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1 Introduction

The random geometric graphs $G(\mathcal{X}, r)$ have been well studied in the last decade; see the monograph [10] and references therein. In order to investigate the typical vertex degree of $G(\mathcal{X}_n, r_n)$, Penrose([11]) defined an empirical process of k_n-nearest neighbor distances in $\mathcal{X}_n$, and showed the weak convergence of the finite-dimensional distributions of that process, scaled and centered, to a Gaussian limit process. He further considered the case $k_n = k$ fixed in [10] later. Given a finite point set $\mathcal{X} \in \mathbb{R}^d$ and given $x \in \mathcal{X}$, the k-nearest neighbor distance means the distance from x to its k-nearest neighbor in $\mathcal{X}$. In the geometric setting, the k-nearest neighbor distance is often a suitable vehicle to deal with degree-related properties of spatial point configurations. It is also closely concerned with k-spacing in statistical testing [14].

In this paper we extend the method of Penrose and establish results analogous to the ones mentioned above for in/out-degree of random geometric digraphs. Theorem 3 shows that the degree distribution of random geometric digraphs in the thermodynamic regime can be either homogeneous or inhomogeneous according to different underlying distributions of point processes. In particular, the degree distribution is Poisson-like when points are uniformly scattered, reminiscent of that of Erdös-Rényi random graphs; otherwise the degree distribution is highly skew, similar with that of many large real-world graphs [6].

Let $\mathcal{X}_n = \{X_1, X_2, \cdots, X_n\}$, $\{X_i\}$ are i. i. d. random variables in $\mathbb{R}^d$ with distribution F having a specified bounded density function f. Let $\mathcal{P}_n = \{X_1, X_2, \cdots, X_{N_n}\}$, $N_n \sim Poi(n)$. So $\mathcal{P}_n$ is a Poisson point process with intensity nf, coupled with $\mathcal{X}_n$. Let $\mathcal{H}_\lambda$ be a homogeneous Poisson process with intensity λ on $\mathbb{R}^d$ and $\|\cdot\|$ be l^2 norm on $\mathbb{R}^d$. Standard random geometric graphs $G(\mathcal{X}_n, r_n)$, $G(\mathcal{P}_n, r_n)$ are defined as in [10], that is, $G(\mathcal{X}_n, r_n)$ (or $G(\mathcal{P}_n, r_n)$) has vertex-set $\mathcal{X}_n$ (or $\mathcal{P}_n$) and an edge X_iX_j $(i \neq j)$ if $||X_i - X_j|| < r_n$. We always assume that $r_n \to 0$ as $n \to \infty$. We now define random geometric digraph models to use in this paper as follows:

Definition 1. $(d = 2)$ *Let* $\alpha \in (0, 2ß]$ *be fixed. Let* $\mathcal{Y}_n = \{Y_1, Y_2, \cdots, Y_n\}$ *be i.i.d. random variables, taking values in* $[0, 2ß)$*, with density function* g*. Associate every point* $X_i \in \mathcal{X}_n$ *a sector, which is centered at* X_i*, with radius* r_n*, amplitude* α *and elevation* Y_i *with respect to the* x*-axis horizontal direction anticlockwise. This sector is denoted as* $S(X_i, Y_i, r_n)$*. We denote by* $G_\alpha(\mathcal{X}_n, \mathcal{Y}_n, r_n)$ *(abbreviated as* G_n*) the digraph with vertex set* $\mathcal{X}_n$*, and with arc* (X_i, X_j)*,* $i \neq j$*, present if and only if* $X_j \in S(X_i, Y_i, r_n)$*. We can define a Poisson version* $G_\alpha(\mathcal{P}_n, \mathcal{Y}_{N_n}, r_n)$ *(*G'_n *for short) similarly.*

In what follows, we will take $g = \frac{1}{2ß}1_{[0,2ß)}$, that is, $Y_i \sim U[0, 2ß)$. Actually, the above model has been first introduced in [3] under the name "random scaled sector graph", with n points uniformly distributed in $[0, 1]^2$. This is an important variant of random geometric graph, and it is used to analyze the performance of wireless sensor networks communicating through optical devices or directional antennae, which are significant in mobile communication[3, 4, 5].

2 Statement of Main Results

We will consider two asymptotic regimes. First, take $k_n \equiv k \in \mathbb{N}$. Second, let $k_n \to \infty$, and

$$\lim_{n\to\infty} \frac{k_n}{\sqrt{n}} = 0. \tag{1}$$

Notice that if we want the sequence $\{k_n\}_{n\geq 1}$ to converge as n tends to infinity, then the above two cases are only choices (and (1) is technically needed in the proofs). In the first regime, define $r_n = r_n(t)$ by $nr_n(t)^2 = t$, for $t > 0$, and in the second, define $r_n = r_n(t)$ by $nr_n(t)^2 = s(k_n + t\sqrt{k_n})$, for $s > 0$, $t \in \mathbb{R}$. Here we introduce a tunable parameter t to adjust the areas of sectors and t has nothing to do with "time", though we will study several random processes with t that evolves. Regulating t allows us to tackle the degree sequences in fine details. The reason why we choose such r_n is to ensure a non-degenerate limit, since nr_n^2 is a good measure of average degree, see the appendix A of [9].

Before proceeding, we give some notations to ease statement. For $\lambda > 0$, let $\rho_\lambda(k) := P(Poi(\lambda) = k)$ and for $A \subseteq \mathbb{Z}^+$, let $\rho_\lambda(A) := P(Poi(\lambda) \in A)$. For $x \in \mathbb{R}^2$, let ϕ, Φ be the density and distribution function of standard normal variables. Given $x \in \mathbb{R}^2$, define $B(x, r)$ the disk with center x and radius r, and let $B_n(x, t) := B(x, r_n(t))$, $S_n(x, y, t) := S(x, y, r_n(t))$ in both limit regimes. Set $\mathcal{X}^x := \mathcal{X} \cup \{x\}$, if $\mathcal{X}$ is a finite set in $\mathbb{R}^2$ and $x \in \mathbb{R}^2$. Denote by $\#\mathcal{X}$ the number of elements in $\mathcal{X}$ and $\mathcal{X}(A) := \#(\mathcal{X} \cap A)$ for $A \subseteq \mathbb{R}^2$.

In the rest of the paper $f_{\max}$ will denote the essential supremum of the probability density function f, i.e. $f_{\max} := \sup\{u : |\{x : f(x) > u\}| > 0\}$. Here and in the rest of the paper $|\cdot|$ denotes Lebesgue measure. We assume $f_{\max} < \infty$ throughout the paper. Next, define the level set when $k_n \to \infty$ as $L_s := \{x \in \mathbb{R}^2 | sf(x) = \frac{2}{\alpha}\}$ and let $L_s^+ := \{x \in \mathbb{R}^2 | sf(x) > \frac{2}{\alpha}\}$. We also put a mild restriction on density function f: let $R := \big\{x \in \mathbb{R}^2 | f(x) > 0, \limsup_{y \longrightarrow x} \frac{|f(y)-f(x)|}{\|y-x\|} < K\big\}$ with some $K < \infty$, and we always assume $F(R) = 1$.

For Borel set $A \subseteq \mathbb{R}^2$, define $\xi_n^{out}(t, A)$, $\xi_n^{\prime out}(t, A)$ be the number of vertices in A of out-degrees at least k_n of G_n and G'_n respectively. More specifically,

$$\xi_n^{out}(t, A) = \sum_{i=1}^{n} 1_{[\mathcal{X}_n(S_n(X_i, Y_i, t)) \geq k_n + 1] \bigcap [X_i \in A]}$$

$$\xi_n^{\prime out}(t, A) = \sum_{i=1}^{N_n} 1_{[\mathcal{P}_n(S_n(X_i, Y_i, t)) \geq k_n + 1] \bigcap [X_i \in A]}$$

Similarly, for in-degree we have,

$$\xi_n^{in}(t, A) = \sum_{i=1}^{n} 1_{[\#\{X_j \in \mathcal{X}_n | X_i \in S_n(X_j, Y_j, t)\} \geq k_n + 1] \bigcap [X_i \in A]}$$

$$\xi_n^{\prime in}(t, A) = \sum_{i=1}^{N_n} 1_{[\#\{X_j \in \mathcal{P}_n | X_i \in S_n(X_j, Y_j, t)\} \geq k_n + 1] \bigcap [X_i \in A]}$$

Notice for the case $k_n \to \infty$, s is suppressed in the above expressions. Also, let $\xi_n^{out}(t) := \xi_n^{out}(t, \mathbb{R}^2)$ etc. for convenience.

The following two lemmas are intermediate steps to prove Theorem 1 and 2. They can be treated in parallel with Theorem 4.12 and 4.13 in [10] through a dependency graph argument.

Lemma 1. *Suppose that $k_n = k$ is fixed, and that A is a Borel set in $\mathbb{R}^2$. The finite-dimensional distributions of the process*

$$n^{-\frac{1}{2}}[\xi_n'^{out}(t,A) - E\xi_n'^{out}(t,A)] \quad , \quad t \geq 0$$

converge to those of a centered Gaussian process $(\xi_\infty'^{out}(t,A), t > 0)$ with covariance $E[\xi_\infty'^{out}(t,A)\xi_\infty'^{out}(u,A)]$ given by

$$\int_A \rho_{\frac{\alpha}{2}tf(x)}([k,\infty))f(x)\mathrm{d}x + \frac{1}{4ß^2}\int_0^{2ß}\int_0^{2ß}\int_A\int_{\mathbb{R}^2}\psi_\infty^{out}(z, f(x_1), y_1, y_2)f^2(x_1)\mathrm{d}z\mathrm{d}x_1\mathrm{d}y_1\mathrm{d}y_2$$

with

$$\begin{aligned}\psi_\infty^{out}(z,\lambda,y_1,y_2) =& P(\{\mathcal{H}_\lambda^z(S(0,y_1,t^{\frac{1}{2}})) \geq k\} \cap \{\mathcal{H}_\lambda^0(S(z,y_2,u^{\frac{1}{2}})) \geq k\}) \\ & - P(\mathcal{H}_\lambda(S(0,y_1,t^{\frac{1}{2}})) \geq k)P(\mathcal{H}_\lambda(S(z,y_2,u^{\frac{1}{2}})) \geq k)\end{aligned}$$

The finite-dimensional distributions of the process

$$n^{-\frac{1}{2}}[\xi_n'^{in}(t,A) - E\xi_n'^{in}(t,A)] \quad , \quad t \geq 0.$$

converge to those of a centered Gaussian process $(\xi_\infty'^{in}(t,A), t > 0)$ with covariance $E[\xi_\infty'^{in}(t,A)\xi_\infty'^{in}(u,A)]$ given by

$$\int_A \rho_{\frac{\alpha}{2}tf(x)}([k,\infty))f(x)\mathrm{d}x + \int_A\int_{\mathbb{R}^2}\psi_\infty^{in}(z, \frac{\alpha}{2ß}f(x_1))f^2(x_1)\mathrm{d}z\mathrm{d}x_1$$

with

$$\begin{aligned}\psi_\infty^{in}(z,\lambda) =& P(\{\mathcal{H}_\lambda^z(B(0,t^{\frac{1}{2}})) \geq k\} \cap \{\mathcal{H}_\lambda^0(B(z,u^{\frac{1}{2}})) \geq k\}) \\ & - P(\mathcal{H}_\lambda(B(0,t^{\frac{1}{2}})) \geq k)P(\mathcal{H}_\lambda(B(z,u^{\frac{1}{2}})) \geq k)\end{aligned}$$

Let $\mathcal{W}$ denote homogeneous white noise of intensity $ß^{-1}$ on $\mathbb{R}^2$, that is, a centered Gaussian process indexed by bounded Borel sets in $\mathbb{R}^2$, with covariance $\mathrm{Cov}(\mathcal{W}(A), \mathcal{W}(B)) = \frac{1}{ß}|A \cap B|$, where $|\cdot|$ as mentioned before is Lebesgue measure. Also, let $\mathcal{W}'$ denote homogeneous white noise of intensity $\frac{2}{\alpha}$.

Lemma 2. *Suppose that $k_n \to \infty$, that (1) holds, and that A is a Borel set in $\mathbb{R}^2$. Let $s > 0$ and suppose $F(A \cap L_s) > 0$. The finite-dimensional distributions of the process*

$$(nk_n)^{-\frac{1}{2}}[\xi_n'^{out}(t,A) - E\xi_n'^{out}(t,A)] \quad , \quad t \in \mathbb{R}$$

converge to those of a centered Gaussian process $(\xi_\infty^{\prime out}(t,A), t \in \mathbb{R})$ *with covariance* $E[\xi_\infty^{\prime out}(t,A)\xi_\infty^{\prime out}(u,A)]$ *given by*

$$\frac{|L_s \cap A|}{s(ß\alpha)^2}\int_0^{2ß}\int_0^{2ß}\int_{\mathbb{R}^2}\mathrm{Cov}(1_{[\mathcal{W}'(S(0,y_1,1))\le t]}, 1_{[\mathcal{W}'(S(z,y_2,1))\le u]})\mathrm{d}z\mathrm{d}y_1\mathrm{d}y_2$$

The finite-dimensional distributions of the process

$$(nk_n)^{-\frac{1}{2}}[\xi_n^{\prime in}(t,A) - E\xi_n^{\prime in}(t,A)] \quad , \quad t \in \mathbb{R}$$

converge to those of a centered Gaussian process $(\xi_\infty^{\prime in}(t,A), t \in \mathbb{R})$ *with covariance* $E[\xi_\infty^{\prime in}(t,A)\xi_\infty^{\prime in}(u,A)]$ *given by*

$$\frac{4\cdot|L_s\cap A|}{s\alpha^2}\int_{\mathbb{R}^2}\mathrm{Cov}(1_{[\mathcal{W}(B(0,1))\le t]}, 1_{[\mathcal{W}(B(z,1))\le u]})\mathrm{d}z.$$

Now we are ready to state our main results.

Theorem 1. *Suppose that* $k_n = k$ *is fixed. The finite-dimensional distributions of the process*

$$n^{-\frac{1}{2}}[\xi_n^{out}(t) - E\xi_n^{out}(t)] \quad , \quad t \ge 0$$

converge to those of a centered Gaussian process $(\xi_\infty^{out}(t), t > 0)$ *with*

$$E[\xi_\infty^{out}(t)\xi_\infty^{out}(u)] = E[\xi_\infty^{\prime out}(t)\xi_\infty^{\prime out}(u)] - h(t)h(u),$$

where

$$h(t) = \int_{\mathbb{R}^2}\left\{\rho_{\frac{\alpha}{2}tf(x)}(k-1)\frac{\alpha}{2}tf(x) + \rho_{\frac{\alpha}{2}tf(x)}([k,\infty))\right\}f(x)\mathrm{d}x \qquad (2)$$

The above result also holds in the case where the superscripts 'out' are replaced by 'in' everywhere.

Theorem 2. *Suppose that* $k_n \to \infty$*, and (1) holds. Let* $s > 0$ *and suppose* $F(L_s) > 0$*. The finite-dimensional distributions of the process*

$$(nk_n)^{-\frac{1}{2}}[\xi_n^{out}(t) - E\xi_n^{out}(t)] \quad , \quad t \in \mathbb{R}$$

converge to those of a centered Gaussian process $(\xi_\infty^{out}(t), t \in \mathbb{R})$ *with*

$$E[\xi_\infty^{out}(t)\xi_\infty^{out}(u)] = E[\xi_\infty^{\prime out}(t)\xi_\infty^{\prime out}(u)] - g(t)g(u),$$

where $g(t) = \phi(t)F(L_s)$*. The above result also holds in the case where the superscripts 'out' are replaced by 'in' everywhere.*

To deal with the degree distribution, let $\eta_n^{out}(t,A)$ and $\eta_n^{in}(t,A)$ be the number of vertices in A of out-degree and in-degree k fixed in G_n respectively.

Theorem 3. *Suppose A is a Borel set in $\mathbb{R}^2$ and $\alpha \geq$ ß. If either $k_n = k$ fixed, or $k_n \to \infty$ and $n^{-1}k_n^2 \ln n \to 0$, then*

$$\lim_{n\to\infty} n^{-1}\xi_n^{out}(t, A) - E[n^{-1}\xi_n^{out}(t, A)] = 0 \qquad a.e. \tag{3}$$

Moreover,

$$\lim_{n\to\infty} n^{-1}\eta_n^{out}(t, A) = \int_A \rho_{\frac{\alpha}{2}tf(x)}(k)f(x)\mathrm{d}x \qquad a.e. \tag{4}$$

The above result also holds in the case where the superscripts 'out' are replaced by 'in' everywhere.

We take expectation on both sides of (4), and let

$$p(k) := E\lim_{n\to\infty} n^{-1}\eta_n^{out}(t, \mathbb{R}^2),$$

so the out-/in-degree distribution of $G_\alpha(\mathcal{X}_n, \mathcal{Y}_n, r_n(t))$, where $nr_n(t)^2 = t$, is

$$p(k) = \frac{(\frac{\alpha}{2}t)^k}{k!}\int_{\mathbb{R}^2} e^{-\frac{\alpha}{2}tf(x)}f(x)^{k+1}\mathrm{d}x, \qquad k \in \mathbb{N}\cup\{0\} \tag{5}$$

If we take the uniform density function $f(x) = 1_{[0,1]^2}(x)$ in (5), then we see that $p(k) = e^{-\frac{\alpha}{2}t}(\frac{\alpha}{2}t)^k/k!$, $k \geq 0$; that is, the degree distribution is $Poi(\frac{\alpha}{2}t)$.

If we take the standard multivariate normal density function $f(x) := f(x_1, x_2) = (1/2ß)e^{-(x_1^2+x_2^2)/2}$, then through the polar coordinate transformation and integration by parts, we obtain

$$p(k) = (4ß/\alpha t) - e^{-\alpha t/4ß}\sum_{i=0}^{k}(\alpha t/4ß)^{i-1}/i! \quad k \geq 0.$$

It is easy to see that $p(k) \to 0$ as $k \to \infty$; and furthermore, since $p(0) = (4ß/\alpha t)(1 - e^{-\alpha t/4ß})$, $p(0) \to 1$ as $t \to 0$ and $p(0) \to 0$ as $t \to \infty$. These observations allow us presumably adjust the parameter t to get different skew degree distributions especially for small k. However, the degree distribution in (5) has a light tail in contrast to the power law distributions [1] because of the fast decay as k tends to infinity. To be precise, by (5) and Stirling formula,

$$p(k) \leq \frac{(\frac{\alpha}{2}tf_{\max})^k}{k!}\int_{\mathbb{R}^2} f(x)\mathrm{d}x = (1 + o(1)) \cdot \frac{(\alpha tef_{\max})^k}{(2k)^k\sqrt{2ßk}} \ll k^{-\beta}$$

for any $\beta > 0$ as $k \to \infty$.

On the other hand, if we want to find a suitable density function f for a given probability distribution $p(k)$ satisfying $p(k) \geq 0$ and $\sum_{k=0}^{\infty} p(k) = 1$, then we simply solve the equation (5), which is the first kind nonlinear singular Fredholm integral equation [2]. However, only approximation solutions of this kind of equations may be obtained.

3 Proof of Means and Degree Distribution

Proposition 1. *(out-degree) Suppose $A \subseteq \mathbb{R}^2$ is a Borel set. If $k_n = k$ is fixed, then*

$$\lim_{n\to\infty} n^{-1}E[\xi_n^{out}(t,A)] = \int_A \rho_{\frac{\alpha}{2}tf(x)}([k,\infty))f(x)\mathrm{d}x$$

If $k_n \to \infty$, and (1) holds, then

$$\lim_{n\to\infty} n^{-1}E[\xi_n^{out}(t,A)] = F(L_s^+ \cap A) + \Phi(t)F(L_s \cap A)$$

Proposition 2. *(in-degree) The same results hold when replace superscripts "out" by "in" in Proposition 1.*

See [15] for the proofs of Proposition 1 and 2.

Proof of Theorem 3. Define a σ filtration: $\mathcal{F}_0 = \{\emptyset, \Omega\}$, and for $1 \le i \le n$, $\mathcal{F}_i = \sigma\{(X_1,Y_1),(X_2,Y_2),\cdots,(X_i,Y_i)\}$.

For out-degree, $\xi_n^{out}(t,A) - E[\xi_n^{out}(t,A)] = \sum_{i=1}^n M_{i,n}^{out}$, with

$$M_{i,n}^{out} = E[\xi_n^{out}(t,A)|\mathcal{F}_i] - E[\xi_n^{out}(t,A)|\mathcal{F}_{i-1}].$$

Let $\xi_{n,i}^{out}(t,A)$ be the number of vertices in A of $G(\mathcal{X}_{n+1}\backslash\{X_i\}, \mathcal{Y}_{n+1}\backslash\{Y_i\}, r_n)$ having out-degree at least k_n. Thereby, $M_{i,n}^{out} = E[\xi_n^{out}(t,A) - \xi_{n,i}^{out}(t,A)|\mathcal{F}_i]$.

We now claim that: For finite set $\mathcal{X} \subseteq \mathbb{R}^2$ and $x \in \mathcal{X}$, there are at most $8k$ points $z \in \mathcal{X}$ having x as their $(\le k) - th$ nearest neighbor, for any $k \in \mathbb{N}$. Here x is the $k-th$ nearest neighbor of z in $\mathcal{X}$ means if we order quantities $\{||w - z|| : w \in \mathcal{X}\backslash\{z\}\}$ increasingly, then $||x - z||$ will be the $k - th$ item in this sequence. Proof. We take a cone with vertex x, central angle ß/4. It's easy to see that there are at most k_n points of $\mathcal{X}$ having x as their $(\le k) - th$ nearest neighbor, since we may look for these points from near to far. The claim follows since the plane is covered by 8 such cones.

Therefore,

$$\begin{aligned} |\xi_n^{out}(t,A) - \xi_{n,i}^{out}(t,A)| &\le |\xi_n^{out}(t,A) - \tilde{\xi}_{n+1}^{out}(t,A)| + |\tilde{\xi}_{n+1}^{out}(t,A) - \xi_{n,i}^{out}(t,A)| \\ &\le (8k_n+1) + (8k_n+1) \le 18k_n, \end{aligned}$$

where let $\tilde{\xi}_{n+1}^{out}(t,A)$ denote the number of vertices in A of out-degrees at least k_n of $G(\mathcal{X}_{n+1}, \mathcal{Y}_{n+1}, r_n)$. Then $|M_{i,n}^{out}| \le 18k_n$. For $\varepsilon > 0$, by Azuma inequality[1],

$$P[|\xi_n^{out}(t,A) - E[\xi_n^{out}(t,A)]| > \varepsilon n] \le 2e^{-\varepsilon^2 n^2/648nk_n^2}.$$

The Borel-Cantelli lemma gives (3). The in-degree case can be proved similarly.

To prove (4), we notice

$$\eta_n^{out}(t,A) = \sum_{i=1}^n 1_{[\mathcal{X}_n(S_n(X_i,Y_i,t))\ge k_n+1]\bigcap[X_i\in A]} - \sum_{i=1}^n 1_{[\mathcal{X}_n(S_n(X_i,Y_i,t))\ge k_n+2]\bigcap[X_i\in A]}$$

and by (3) and the proof of Proposition 1, the result follows immediately. The in-degree case also follows similarly. □

4 Proof of Central Limit Theorems

In this section we first develop some moments for non-Poisson case in the limit regime $k_n \to \infty$, which is crucial to de-Poisson Lemma 1 and 2.

For $n, m \in \mathbb{N}$, set

$$T^{out}_{m,n}(t) := \sum_{i=1}^{m} 1_{[\mathcal{X}_m(S_n(X_i,Y_i,t)\setminus\{X_i\})\geq k_n]}$$

and

$$T^{in}_{m,n}(t) := \sum_{i=1}^{m} 1_{[\#\{X_j\in\mathcal{X}_m\setminus\{X_i\}|X_i\in S_n(X_j,Y_j,t)\}\geq k_n]}$$

Then we see $T^{out}_{n,n}(t) = \xi^{out}_n(t), T^{out}_{N_n,n}(t) = \xi'^{out}_n(t)$ and $T^{in}_{n,n}(t) = \xi^{in}_n(t), T^{in}_{N_n,n}(t) = \xi'^{in}_n(t)$. Set $\tilde{D}^{out}_{m,n}(t) := T^{out}_{m+1,n}(t) - T^{out}_{m,n}(t)$, then $\tilde{D}^{out}_{m,n}(t) = D^{out}_{m,n}(t) + \hat{D}^{out}_{m,n}(t)$, where

$$D^{out}_{m,n}(t) = \sum_{i=1}^{m} 1_{[\mathcal{X}_m(S_n(X_i,Y_i,t)\setminus\{X_i\})=k_n-1]\cap[X_{m+1}\in S_n(X_i,Y_i,t)]}$$

$$\hat{D}^{out}_{m,n}(t) = 1_{[\mathcal{X}_m(S_n(X_{m+1},Y_{m+1},t))\geq k_n]}$$

Set $\tilde{D}^{in}_{m,n}(t) := T^{in}_{m+1,n}(t) - T^{in}_{m,n}(t)$, then $\tilde{D}^{in}_{m,n}(t) = D^{in}_{m,n}(t) + \hat{D}^{in}_{m,n}(t)$, where

$$D^{in}_{m,n}(t) = \sum_{i=1}^{m} 1_{[\#\{X_j\in\mathcal{X}_m\setminus\{X_i\}|X_i\in S_n(X_j,Y_j,t)\}=k_n-1]\cap[X_i\in S_n(X_{m+1},Y_{m+1},t)]}$$

$$\hat{D}^{in}_{m,n}(t) = 1_{[\#\{X_j\in\mathcal{X}_m|X_{m+1}\in S_n(X_j,Y_j,t)\}\geq k_n]}$$

Lemma 3. *Suppose $k_n \to \infty$ and (1) holds. Then*

$$\lim_{n\to\infty} \sup_{\{m||m-n|\leq n^{2/3}\}} |k_n^{-1/2} E\tilde{D}^{out}_{m,n}(t) - \phi(t)F(L_s)| = 0$$

The same formula holds when replace superscript "out" by "in".

Lemma 4. *Suppose $k_n \to \infty$ and (1) holds. Then*

$$\lim_{n\to\infty} \sup_{n-n^{2/3}\leq l<m\leq n+n^{2/3}} |k_n^{-1} E\tilde{D}^{out}_{l,n}(t)\tilde{D}^{out}_{m,n}(u) - \phi(t)\phi(u)F(L_s)^2| = 0$$

The same formula holds when replace superscripts "out" by "in".

Lemma 5. *Suppose $k_n \to \infty$ and (1) holds. Let $t, u \in \mathbb{R}$. Then*

$$\limsup_{n\longrightarrow\infty} \left(k_n^{-3/2} \cdot \sup_{\{m||m-n|\leq n^{2/3}\}} E[\tilde{D}^{out}_{m,n}(t)^2]\right) < \infty.$$

The same formula holds when replace superscript "out" by "in".

The readers are referred to [15] for more details. To prove Theorem 1 and 2, we will use de-Poisson techniques, which are given in [8, 12] and later generalized in [10, 13]. We will also need Cramér-Wold device[7].

Proof of Theorem 1. Let $M \in \mathbb{N}$, $B = (b_1, \cdots, b_M) \in \mathbb{R}^M$, $T = (t_1, \cdots, t_M) \in (0, \infty)^M$.

For out-degree, $\mathcal{X} \subset \mathbb{R}^2, \mathcal{Y} \subset [0, 2ß)$ with $\text{card}(\mathcal{X}) = \text{card}(\mathcal{Y})$, set

$$H_0(\mathcal{X}, \mathcal{Y}) := \sum_{i=1}^{M} \sum_{(x,y)\in(\mathcal{X},\mathcal{Y})} b_i 1_{[\mathcal{X}(S(x,y,t_i^{1/2}))\geq k_n+1]}$$

and let $H_n(\mathcal{X}, \mathcal{Y}) = H_0(n^{1/2}\mathcal{X}, \mathcal{Y})$. $(x, y) \in \mathbb{R}^2 \times [0, 2ß) \subset \mathbb{R}^3$. Set $\xi_n^{\prime out}(T, B, A) := \sum_{m=1}^{M} b_m \xi_n^{\prime out}(t_m, A)$ and $\text{Var}\big(\xi_n^{\prime out}(T, B, A)\big) := \sigma^{\prime out}(T, B, A)$, we have $H_n(\mathcal{P}_n, \mathcal{Y}_{N_n}) = \xi_n^{\prime out}(T, B, \mathbb{R}^2)$, and what's more, $(\mathcal{P}_n, \mathcal{Y}_{N_n})$ is a $3-$dimensional Poisson process, which may be coupled with $(\mathcal{X}_n, \mathcal{Y}_n)$ in the same way as $\mathcal{P}_n$ does with $\mathcal{X}_n$. By Lemma 1, $n^{-1/2}(H_n(\mathcal{P}_n, \mathcal{Y}_{N_n}) - EH_n(\mathcal{P}_n, \mathcal{Y}_{N_n})) \xrightarrow{\text{D}} \mathcal{N}(0, \sigma^{\prime out}(T, B, \mathbb{R}^2))$. Let $\mathcal{H}_\lambda$ be a $3-$dimensional homogeneous Poisson process and denote point $(x, y) \in \mathbb{R}^2 \times \mathbb{R}$, $\mathcal{H}_\lambda := (\mathcal{H}_\lambda^{(1)}, \mathcal{H}_\lambda^{(2)})$ with $x \in \mathcal{H}_\lambda^{(1)}, y \in \mathcal{H}_\lambda^{(2)}$. Next, we say $H_0(\mathcal{X}, \mathcal{Y})$ is strongly stabilizing on $\mathcal{H}_\lambda$ if there are $a.s.$ finite random variables T and $\Delta(\mathcal{H}_\lambda)$ such that with probability 1, $\Delta(A) = \Delta(\mathcal{H}_\lambda)$ for all finite $A := (A_1, A_2) \subset \mathbb{R}^2 \times [0, 2ß)$ with $\text{card}(A_1) = \text{card}(A_2)$, satisfying $A \cap (B(0, T) \times [0, 2ß)) = \mathcal{H}_\lambda \cap (B(0, T) \times [0, 2ß))$. Here, $\triangle(\mathcal{H}_\lambda) := H_0(\mathcal{H}_\lambda^0) - H_0(\mathcal{H}_\lambda)$. Thus, H_0 is strongly stable since it has finite range. We have

$$\begin{aligned} E[\triangle(\mathcal{H}_\lambda)] &= E[H_0(\mathcal{H}_\lambda^0) - H_0(\mathcal{H}_\lambda)] \\ &= E\Big[\sum_{i=1}^{M} b_i\Big(\sum_{(x,y)\in\mathcal{H}_\lambda} 1_{[\mathcal{H}_\lambda^{(1),0}(S(x,y,t_i^{1/2}))\geq k+1]} + 1_{[\mathcal{H}_\lambda^{(1),0}(S(0,0,t_i^{1/2}))\geq k+1]}\Big) \\ &\quad - \sum_{i=1}^{M} b_i\Big(\sum_{(x,y)\in\mathcal{H}_\lambda} 1_{[\mathcal{H}_\lambda^{(1)}(S(x,y,t_i^{1/2}))\geq k+1]}\Big)\Big] \\ &= E\sum_{i=1}^{M} b_i\Big(1_{[Poi(2ß\lambda\cdot\frac{\alpha t_i}{2})\geq k]} + \sum_{\substack{(x,y)\in\mathcal{H}_\lambda \\ 0\in S(x,y,t_i^{1/2})}} 1_{[\mathcal{H}_\lambda^{(1)}(S(x,y,t_i^{1/2}))=k]}\Big) \\ &= \sum_{i=1}^{M} b_i\Big(\rho_{\lambda ß\alpha t_i}([k, \infty)) + \lambda 2ß \cdot \frac{\alpha t_i}{2}(k-1)\Big). \end{aligned}$$

By (2) and the Cox process $\mathcal{H}_{\varphi(X,Y)}$ with $\varphi(X, Y) := (1/2ß)f(X)$, we have $E[\triangle(\mathcal{H}_{\varphi(X,Y)})] = \sum_{i=1}^{M} b_i h(t_i)$. Set $t_{\max} = \max\{t_1, \cdots, t_M\}$, we have $|H_n(\mathcal{X}_m, \mathcal{Y}_m)| \leq m \sum_{i=1}^{M} |b_i|$ and $|H_n(\mathcal{X}_{m+1}, \mathcal{Y}_{m+1}) - H_n(\mathcal{X}_m, \mathcal{Y}_m)| \leq \big(\sum_{i=1}^{M} b_i\big)\cdot$

$\big[\#\{X_i\in\mathcal{X}_m|X_{m+1}\in S_n(X_i,Y_i,t_{\max})\}+1\big]$, which is stochastically dominated by $c\cdot[Bin(m,f_{\max}ßr_n(t_{\max})^2)+1]$ having a uniformly bounded fourth moment when $m\le 2n$. Therefore by a simple variant of Theorem 2.16([10]) to a marked point process [13] (in particular the translation-invariance of $H_0(\mathcal{X},\mathcal{Y})$ is only required for $\mathcal{X}$), $n^{-1/2}(H_n(\mathcal{X}_n,\mathcal{Y}_n)-EH_n(\mathcal{X}_n,\mathcal{Y}_n))\xrightarrow{\mathrm{D}}\mathcal{N}(0,\tau_{out}^2)$ with $\tau_{out}^2:=\sigma'^{out}(T,B,\mathbb{R}^2)-(E[\triangle(\mathcal{H}_{\varphi(X,Y)})])^2$. The first part of the theorem then follows by Cramér-Wold device.

For in-degree, let $\mathcal{X}\subset\mathbb{R}^2,\mathcal{Y}\subset[0,2ß)$ with $\mathrm{card}(\mathcal{X})=\mathrm{card}(\mathcal{Y})$, and the elements $(x,y)\in(\mathcal{X},\mathcal{Y})$ be ordered pairs. Reset

$$H_0(\mathcal{X},\mathcal{Y}):=\sum_{i=1}^{M}\sum_{x'\in\mathcal{X}}b_i1_{[\#\{x\in\mathcal{X}|x'\in S(x,y,t_i^{1/2})\}\ge k_n+1]}$$

and let $H_n(\mathcal{X},\mathcal{Y})=H_0(n^{1/2}\mathcal{X},\mathcal{Y})$. Set $\xi_n'^{in}(T,B,A):=\sum_{m=1}^{M}b_m\xi_n'^{in}(t_m,A)$ and $\mathrm{Var}\big(\xi_n'^{in}(T,B,A)\big):=\sigma'^{in}(T,B,A)$, we have $H_n(\mathcal{P}_n,\mathcal{Y}_{N_n})=\xi_n'^{in}(T,B,\mathbb{R}^2)$, and $(\mathcal{P}_n,\mathcal{Y}_{N_n})$ is a $3-$dimensional Poisson process coupled with $(\mathcal{X}_n,\mathcal{Y}_n)$. By Lemma 1, $n^{-1/2}(H_n(\mathcal{P}_n,\mathcal{Y}_{N_n})-EH_n(\mathcal{P}_n,\mathcal{Y}_{N_n}))\xrightarrow{\mathrm{D}}\mathcal{N}(0,\sigma'^{in}(T,B,\mathbb{R}^2))$. Also, H_0 is strongly stable. Let $\mathcal{H}_\lambda$ be a $3-$dimensional homogeneous Poisson process and $\mathcal{H}_\lambda:=(\mathcal{H}_\lambda^{(1)},\mathcal{H}_\lambda^{(2)})$ as above. Then

$$\begin{aligned}E[\triangle(\mathcal{H}_\lambda)]&=E[H_0(\mathcal{H}_\lambda^0)-H_0(\mathcal{H}_\lambda)]\\&=E\Big[\sum_{i=1}^{M}b_i\Big(\sum_{x'\in\mathcal{H}_\lambda^{(1)}}1_{[\#\{x\in\mathcal{H}_\lambda^{(1),0}|x'\in S(x,y,t_i^{1/2})\}\ge k+1]}\\&\quad+1_{[\#\{x\in\mathcal{H}_\lambda^{(1),0}|0\in S(x,y,t_i^{1/2})\}\ge k+1]}\Big)\\&\quad-\sum_{i=1}^{M}b_i\Big(\sum_{x'\in\mathcal{H}_\lambda^{(1)}}1_{[\#\{x\in\mathcal{H}_\lambda^{(1)}|x'\in S(x,y,t_i^{1/2})\}\ge k+1]}\Big)\Big]\\&=E\sum_{i=1}^{M}b_i\Big(1_{[Poi(2ß\lambda\cdot\frac{\alpha t_i}{2})\ge k]}+\sum_{x'\in\mathcal{H}_\lambda^{(1)}\cap S(0,0,t_i^{1/2})}1_{[\#\{x\in\mathcal{H}_\lambda^{(1)}|x'\in S(x,y,t_i^{1/2})\}=k]}\Big)\\&=\sum_{i=1}^{M}b_i\Big(\rho_{\lambda ß\alpha t_i}([k,\infty))+2ß\lambda\cdot\frac{\alpha t_i}{2}(k-1)\Big).\end{aligned}$$

The remain proof is similar with the out-degree case. □

Proof of Theorem 2. Let T and $B\in\mathbb{R}^M$.

For out-degree, $\mathcal{X}\subset\mathbb{R}^2,\mathcal{Y}\subset[0,2ß)$ with $\mathrm{card}(\mathcal{X})=\mathrm{card}(\mathcal{Y})$, set

$$H_n(\mathcal{X},\mathcal{Y}):=k_n^{-1/2}\sum_{i=1}^{M}\sum_{(x,y)\in(\mathcal{X},\mathcal{Y})}b_i1_{[\mathcal{X}(S_n(x,y,t_i))\ge k_n+1]}$$

By Lemma 2, we have $H_n(\mathcal{P}_n, \mathcal{Y}_{N_n}) = k_n^{-1/2} \xi_n'^{out}(T, B, \mathbb{R}^2)$ and $n^{-1/2}(H_n(\mathcal{P}_n, \mathcal{Y}_{N_n}) - EH_n(\mathcal{P}_n, \mathcal{Y}_{N_n})) \xrightarrow{\mathrm{D}} \mathcal{N}(0, \sigma'^{out}(T, B, \mathbb{R}^2))$. Set $\alpha := \sum_{i=1}^{M} b_i \phi(t_i) F(L_s)$, and $R_{m,n}^{out} := H_n(\mathcal{X}_{m+1}, \mathcal{Y}_{m+1}) - H_n(\mathcal{X}_m, \mathcal{Y}_m)$. Therefore, $R_{m,n}^{out} = k_n^{-1/2} \sum_{i=1}^{M} b_i \tilde{D}_{m,n}^{out}(t_i)$. By Lemmas 3, 4 and 5, we have

$$\lim_{n\to\infty} \Big(\sup_{n-n^{2/3} \le m \le n+n^{2/3}} |ER_{m,n}^{out} - \alpha| \Big) = 0$$

$$\lim_{n\to\infty} \Big(\sup_{n-n^{2/3} \le m < m' \le n+n^{2/3}} |E[R_{m,n}^{out} R_{m',n}^{out}] - \alpha^2| \Big) = 0$$

and

$$\lim_{n\to\infty} \Big(n^{-1/2} \sup_{n-n^{2/3} \le m \le n+n^{2/3}} E[(R_{m,n}^{out})^2] \Big) = 0.$$

respectively. Also $|H_n(\mathcal{X}_m, \mathcal{Y}_m)| \le m \sum_{i=1}^{M} |b_i|$. Then Theorem2.12([10]) implies $n^{-1/2} \cdot (H_n(\mathcal{X}_n, \mathcal{Y}_n) - EH_n(\mathcal{X}_n, \mathcal{Y}_n)) \xrightarrow{\mathrm{D}} \mathcal{N}(0, \sigma^{out}(T, B))$, with $\sigma^{out}(T, B) := \sigma'^{out}(T, B, \mathbb{R}^2) - \alpha^2$. Hence, $\sigma^{out}(T, B) = \mathrm{Var} \sum_{i=1}^{M} b_i \xi_\infty^{out}(t_i)$. The first part of the theorem then follows by Cramér-Wold device.

For in-degree, let

$$H_n(\mathcal{X}, \mathcal{Y}) := k_n^{-1/2} \sum_{i=1}^{M} \sum_{x' \in \mathcal{X}} b_i 1_{[\#\{x \in \mathcal{X} | x' \in S_n(x, y, t_i)\} \ge k_n + 1]}$$

We then argue likewise to complete the proof. □

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