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Weighted Composition Operators on Weighted Vector-Valued Bergman Spaces

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Abstract. In this paper we first introduce weighted vector-valued Bergman spaces $\mathcal{A}^p_{X,\lambda}(\Lambda^+)$ of the upper half-plane. We then discuss their basic properties and prove a Carleson type Theorem for these spaces. Finally, we characterize boundedness of weighted composition operators on weighted vector-valued Bergman spaces.

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1. Introduction

Let G be a non-empty set, X a topological vector space, $F(G, X)$ a topological vector space of functions from G to X with point-wise vector space operations and $\varphi : G \longrightarrow G$ be a function such that $fo\varphi \in F(G,X)$ for all $f \in F(G,X)$. Then the linear transformation $C_\varphi : F(G,X) \longrightarrow F(G\,,X)$ defined as $C_\varphi(f) = fo\varphi$ for all $f \in F(G,X)$, is called a composition transformation induced by φ, on the space $F(G\,,X)$. If C_φ is continuous, then it is called a composition operator or substitution operator, induced by φ, on the space $F(G\,,X)$. Further if $\psi: G \longrightarrow \mathbb{C}$ be such that $\psi(fo\varphi) \in F(G,X)$ for all $f \in F(G,X)$, then the linear transformation $W_{\varphi,\psi} : F(G,X) \longrightarrow F(G\,,X)$ defined as $W_{\varphi,\psi}(f) = \psi(fo\varphi)$ is called a weighted composition transformation induced by φ and ψ, on the space $F(G,X)$. If $W_{\varphi,\psi}$ is continuous, then $W_{\varphi,\psi}$ is called a weighted composition operator induced by φ and ψ. If $\psi = 1$, then $W_{\varphi,\psi} = C_\varphi$ and if $\varphi(x) = x$, then $W_{\varphi,\psi}$ is called multiplication operator or transformation (depending upon whether it is continuous or not) and is denoted by M_ψ. For more about composition operators we refer to [1] and [13].

As a consequence of the Littlewood Subordination principle it is known that every analytic self-map φ of the open unit disk $\mathbb{D}$ induces a bounded composition operator on Hardy and weighted Bergman spaces of the open unit disk $\mathbb{D}$ (see [1]). However, if we move to Hardy and weighted Bergman spaces of the upper half-plane $\Lambda^+ = \{(x,y) : x,y \in \mathbb{R}\,, y > 0\}$ the situation is entirely different. There do exist analytic self-maps of the upper half-plane, which do not induce composition operators on the Hardy spaces and weighted Bergman spaces of the upper half-plane. Interesting work on composition operators on Hardy spaces of the upper half-plane have been done by Singh [10], Singh and Sharma [11], [12], Sharma [17] , Matache [6], [7] and Sharma, Sharma and Shabir [18] and [19]. In this paper, we characterize weighted composition operators on weighted vector-valued Bergman spaces of the upper half-plane.

Recently several authors have studied weighted composition operators on different spaces of analytic functions. (see for example [2], [3], [4], [5], [8], [9], [11], [14], [15] and references there in for more details).

The plan of the rest of the paper is as follows. In the next section we introduce weighted vector-valued Bergman spaces of the upper half-plane. Section 3 is devoted to Carleson type Theorem of the weighted vector-valued Bergman spaces of the upper half-plane. The fourth section deals with the boundedness of weighted composition operators on the weighted vector-valued Bergman spaces of the upper half-plane.

2. Weighted vector–valued Bergman spaces of the upper half-plane

In this section, weighted vector-valued Bergman spaces $\mathcal{A}^p_{X,\lambda}(\Lambda^+)$ of the upper half-plane are introduced and their basic properties are discussed.

Let $(X,\ \|.\|_X)$ be a complex Banach space and $f : \Lambda^+ \longrightarrow X$ be a map. Then f is said to be analytic if $x^* o f : \Lambda^+ \longrightarrow \mathbb{C}$ is analytic for every $x^* \in X^*$, where X^* is the dual space of X. Let $0 < p < \infty$ and $\lambda \in (-1,\infty)$. Then $\mathcal{L}^p_X(\Lambda^+,\ d\mu_\lambda)$ denotes the collection of all Bochner p −integrable functions $f : \Lambda^+ \longrightarrow X$ for which

$$|||f|||_{p,\lambda} = \left[\int_{\Lambda^+} \|f\|_X^p \, d\mu_\lambda\right]^{1/p} < \infty,$$

where

$$d\mu_\lambda(z) = \frac{1}{\pi}(\lambda+1)\big(2\,Im(z)\big)^{\lambda} d\vartheta(z); \quad d\vartheta(z) = dxdy; \; z = x + iy.$$

We denote

$$\mathcal{A}^p_{X,\lambda}(\Lambda^+) = \{ f \mid f \in \mathcal{L}^p_X(\Lambda^+, d\mu_\lambda) \text{ and } f \text{ is analytic } \}.$$

We call $\mathcal{A}^p_{X,\lambda}(\Lambda^+)$ as weighted vector-valued Bergman space of the upper half-plane Λ^+. If $\lambda = 0$, then $\mathcal{A}^p_{X,\lambda}(\Lambda^+)$ is same as $\mathcal{A}^p_X(\Lambda^+)$, the vector-valued Bergman space of the upper half-plane Λ^+.

For $r \in (0,1)$ and $z = (x,y) \in \Lambda^+$, we define

$$S(z, ry) = \{\omega \in \Lambda^+ : \; |\omega - z| < ry\}.$$

If there exist constants c_1 and c_2 such that $ac_1 \le b \le ac_2$, then we write $a \approx b$.

Lemma 2.1. *Let* $0 < p < \infty$, *-1*$< \lambda < \infty$, $z = (x,y) \in \Lambda^+$. *Then*

$$\|f(z)\|^p_X \le \frac{c\,|||f|||^p_{p,\lambda}}{2^\lambda(\lambda+1)\,y^{\lambda+2}},$$

for all $f \in \mathcal{A}^p_{X,\lambda}(\Lambda^+)$.

Proof: Since $z \longrightarrow \|f(z)\|^p_X$ is sub-harmonic for every analytic function f and $p > 0$, by the area sub-mean value property , we have

$$\|f(z)\|^p_X \le \frac{1}{\vartheta\big(S(z,ry)\big)} \int_{S(z,ry)} \|f(\omega)\|^p_X \, d\vartheta(\omega)$$

$$= \frac{1}{\pi r^2 y^2} \int_{S(z,ry)} \|f(\omega)\|^p_X \, d\vartheta(\omega).$$

Since $Im(\omega) \approx Im(z)$ for $\omega \in S(z,ry)$, we can find a positive constant c such that

$$\|f(z)\|^p_X \le \frac{c}{2^\lambda(\lambda+1) r^2\, y^{\lambda+2}} \int_{S(z,ry)} \|f(\omega)\|^p_X \, d\mu_\lambda(\omega)$$

$$\le \frac{c\,|||f|||^p_{p,\lambda}}{2^\lambda(\lambda+1) r^2\, y^{\lambda+2}}.$$

Letting $r \longrightarrow 1$, we obtain the desired inequality. ■

Lemma 2.1 and the Montel theorem suggest that $\mathcal{A}^p_{X,\lambda}(\Lambda^+)$ is an F-space when $p \in (0,1)$ and a Banach space when $p \in [1,\infty)$. If $(X, \|.\|_X) \equiv (E, \langle .,. \rangle_E)$ is a separable Hilbert space and $p = 2$, then $\mathcal{A}^p_{E,\lambda}(\Lambda^+)$ becomes a Hilbert space under the following definition of the inner product $\langle\langle .,. \rangle\rangle$; $f, g \in \mathcal{A}^p_{E,\lambda}(\Lambda^+)$,

$$\langle\langle f, g \rangle\rangle = \int_{\Lambda^+} \langle f(z), g(z) \rangle_{\mathrm{E}} \, d\mu_\lambda(\mathrm{z}).$$

When $X = \mathbb{C}$, then we drop X from the notation and write simply $\mathcal{A}^p_\lambda(\Lambda^+)$ for $\mathcal{A}^p_{X,\lambda}(\Lambda^+)$ and $\|.\|_{p,\lambda}$ for $|||.|||_{p,\lambda}$.

For each $z \in \Lambda^+$ and $a \in E$, we define the evaluation mapping $\Lambda_z^a : \mathcal{A}^2_{E,\lambda}(\Lambda^+) \longrightarrow \mathbb{C}$ as $\Lambda_z^a(f) = \langle f(z), a\rangle$ for every $f \in \mathcal{A}^2_{E,\lambda}(\Lambda^+)$. By Lemma 2.1, $\Lambda_z^a \in \left(\mathcal{A}^2_{E,\lambda}(\Lambda^+)\right)^*$. Hence $\mathcal{A}^2_{E,\lambda}(\Lambda^+)$ is a vector-valued functional Hilbert space. Thus there exists a unique function $K_\lambda^a(z,.) \in \mathcal{A}^2_{E,\lambda}(\Lambda^+)$ such that

$$\langle f(z), a\rangle = \Lambda_z^a(f) = \langle\langle f, \quad K_\lambda^a(z,.)\rangle\rangle$$

for every $f \in \mathcal{A}^2_{E,\lambda}(\Lambda^+)$. The function $K_\lambda^a(z,.)$ is called vector-valued Bergman kernel.
For any positive integer *n*, let

$$S_n(z) = \sqrt{\frac{2^{\lambda+2}\Gamma(n+2+\lambda)}{n!\ \Gamma(2+\lambda)}}\left(\frac{z-i}{z+i}\right)^n\left(\frac{i}{z+i}\right)^{\lambda+2},$$

$z = (x,y) \in \Lambda^+$. Here, $\Gamma(s)$ stands for the usual gamma function, which is analytic function of 's' in the whole complex plane, except for the simple poles at the points $\{0,-1,-2,-3,\ldots\}$. The family $\{S_n : n \geq 1\}$ is an orthonormal basis for $\mathcal{A}^2_\lambda(\Lambda^+)$. One can also show that if $\{e_n : n \geq 1\}$ is an orthonormal basis for, then the family $\{S_{m,n} : m,n \geq 1\}$, where

$$S_{m,n}(z) = \sqrt{\frac{2^{\lambda+2}\Gamma(n+2+\lambda)}{n!\ \Gamma(2+\lambda)}}\left(\frac{z-i}{z+i}\right)^n\left(\frac{i}{z+i}\right)^{\lambda+2} e_m,$$

forms an orthonormal basis for $\mathcal{A}^2_{E,\lambda}(\Lambda^+)$. When $\lambda = 0$ and $E = \mathbb{C}$, they are just the images of $\sqrt{n+1}\,z^n$ in $\mathcal{A}^2(\Lambda^+)$ under the Cayley's transform.
In the next theorem we obtain a representation for $K_\lambda^a(z,.)$.

Theorem 2.2: *For* $z = (x,y) \in \Lambda^+$ *and* $a \in E$, $K_\lambda^a(z,.)$ *is given by* $K_\lambda^a(z,\omega) = \frac{i^{\lambda+2}}{(z-\overline{\omega})^{\lambda+2}}\, a$, *Moreover,* $|||K_\lambda^a(z,.)|||^2_{2,\lambda} = \frac{\|a\|_E^2}{(2y)^{\lambda+2}}$.

Proof: By Theorem 3.2 in [16], we have

$K_\lambda^a(z,\omega)$

$$= \sum_j\sum_n\sum_m \langle a, e_j\rangle\langle e_j, S_{m,n}(\omega)\rangle S_{m,n}(z)$$

$$= \sum_j\sum_n \frac{2^{\lambda+2}\Gamma(n+2+\lambda)}{n!\ \Gamma(2+\lambda)}\frac{(-i)^{\lambda+2}\ (i)^{\lambda+2}}{(\overline{\omega}-i)^{\lambda+2}\ (z+i)^{\lambda+2}}\left(\frac{\overline{\omega}+i}{\overline{\omega}-i}\right)^n\left(\frac{z-i}{z+i}\right)^n\langle a, e_j\rangle\sum_m\langle e_j, e_m\rangle\, e_m$$

$$= \sum_j \frac{(-i)^{\lambda+2}\ (i)^{\lambda+2}\ (2)^{\lambda+2}}{(\overline{\omega}-i)^{\lambda+2}\ (z+i)^{\lambda+2}}\sum_n\frac{\Gamma(n+2+\lambda)}{n!\ \Gamma(2+\lambda)}\left[\frac{(\overline{\omega}+i)(z-i)}{(\overline{\omega}+i)(z-i)}\right]^n\langle a, e_j\rangle\, e_j$$

$$= \sum_j \frac{(-i)^{\lambda+2}\ (i)^{\lambda+2}\ (2)^{\lambda+2}}{(\overline{\omega}-i)^{\lambda+2}\ (z+i)^{\lambda+2}}\ \frac{\langle a, e_j\rangle\, e_j}{\left[1-\frac{(\overline{\omega}+i)(z-i)}{(\overline{\omega}-i)(z+i)}\right]^{\lambda+2}}$$

$$= \frac{i^{\lambda+2}}{(z-\overline{\omega})^{\lambda+2}}\sum_j\langle a, e_j\rangle\, e_j = \frac{i^{\lambda+2}}{(z-\overline{\omega})^{\lambda+2}}\, a.$$

Also,

$$|||K_\lambda^a(z\,,.)|||_{2,\lambda}^2 = \langle\langle K_\lambda^a(z\,,.), K_\lambda^a(z\,,.)\rangle\rangle = \langle K_\lambda^a(z\,,z)\,,a\rangle$$
$$= \langle \frac{i^{\lambda+2}}{(z-\bar{z})^{\lambda+2}}\,,a\rangle = \frac{\|a\|_E^2}{(2y)^{\lambda+2}},$$

where $y = Im(z)$.

■

Corollary 2.3 : *For* $z = (x, y) \in \Lambda^+$ *, we have*

$$\int_{\Lambda^+} \frac{(2y)^{\lambda+2}}{|z-\overline{\omega}|^{2(\lambda+2)}}\, d\mu_\lambda(\omega) = 1.$$

Proof: For $z = (x, y) \in \Lambda^+$,

$$\int_{\Lambda^+} \frac{(2y)^{\lambda+2}}{|z-\overline{\omega}|^{2(\lambda+2)}}\, d\mu_\lambda(\omega) = \frac{(2y)^{\lambda+2}}{\|a\|_E^2}\int_{\Lambda^+} \frac{1}{(z-\overline{\omega})^{\lambda+2}}\overline{\frac{1}{(z-\overline{\omega})^{\lambda+2}}}\;\|a\|_E^2\, d\mu_\lambda(\omega)$$
$$= \frac{(2y)^{\lambda+2}}{\|a\|_E^2}\int_{\Lambda^+} \frac{1}{(z-\overline{\omega})^{\lambda+2}}\overline{\frac{1}{(z-\overline{\omega})^{\lambda+2}}}\langle a, a\rangle_E\, d\mu_\lambda(\omega)$$
$$= \frac{(2y)^{\lambda+2}}{\|a\|_E^2}\int_{\Lambda^+} \langle K_\lambda^a(z,.), K_\lambda^a(z,.)\rangle_E\, d\mu_\lambda(\omega)$$
$$= \frac{(2y)^{\lambda+2}}{\|a\|_E^2}|||K_\lambda^a(z,.)|||_{2,\lambda}^2$$
$$= 1$$

■

Proposition 2.4 : *For* $z = (x, y) \in \Lambda^+$ *and* $\omega \in S(z\,,ry)$ *, we have*

$$\|K_\lambda^a(z\,,\omega)\|_E \approx \frac{1}{y^{\lambda+2}}\|a\|_E.$$

Proof: For $z = (x,y) \in \Lambda^+$ and $\omega = (t\,,u) \in \Lambda^+$,

$$\|K_\lambda^a(z\,,\omega)\|_E = \frac{\|a\|_E}{|z-\overline{\omega}|^{\lambda+2}}$$
$$= \frac{\|a\|_E}{[(x-t)^2+(y+u)^2]^{\frac{\lambda+2}{2}}}$$
$$\leq \frac{c\,\|a\|_E}{y^{\lambda+2}}.$$

Again if $\omega \in S(z\,,ry)$, then $Im(z) \approx Im(\omega)$, so $|z-\overline{\omega}|^2 \leq c'y^2$. Therefore,

$$\|K_\lambda^a(z\,,\omega)\|_E = \frac{\|a\|_E}{|z-\overline{\omega}|^{\lambda+2}} \geq \frac{c''\|a\|_E}{y^{\lambda+2}},\quad c'' = \frac{1}{\sqrt{c'}}.$$

Thus

$$\frac{c''\|a\|_E}{y^{\lambda+2}} \leq \|K_\lambda^a(z\,,\omega)\|_E \leq \frac{c\,\|a\|_E}{y^{\lambda+2}}.$$

Hence

$$\|K_\lambda^a(z,\omega)\|_E \approx \frac{1}{y^{\lambda+2}}\|a\|_E. \qquad \blacksquare$$

We next show that $K_\lambda^a(z,\omega)$ belong to $\mathcal{A}_{E,\lambda}^p(\Lambda^+)$ for all $1 < p < \infty$.

Proposition 2.5 : *For* $1 < p < \infty$, $-1 < \lambda < \infty$, $z = (x,y) \in \Lambda^+$ *and* $a \in E$, $K_\lambda^a(z,.) \in \mathcal{A}_{E,\lambda}^p(\Lambda^+)$.

Proof: We know that for $x, y \in \mathbf{R}$, $y > 0$

$$P_y(x,t) = \frac{1}{\pi}\frac{y}{(x-t)^2+y^2}$$

is the Poisson kernel for Λ^+ and hence

$$\frac{1}{\pi}\int_{-\infty}^{\infty}\frac{y}{(x-t)^2+y^2}dt = 1.$$

If $z = (x,y) \in \Lambda^+$ and $\omega = (t,u) \in \Lambda^+$, then

$$\int_{\Lambda^+}\|K_\lambda^a(z,\omega)\|_E^p\, d\mu_\lambda(\omega)$$
$$= \|a\|_E^p\int_{\Lambda^+}\frac{1}{|z-\overline{\omega}|^{(\lambda+2)p}}d\mu_\lambda(\omega)$$
$$= \frac{2^\lambda(\lambda+1)\|a\|_E^p}{\pi}\int_0^\infty\int_{-\infty}^\infty\frac{1}{|(x,y)-(t,-u)|^{p(\lambda+2)}}u^\lambda dt du$$
$$= \frac{2^\lambda(\lambda+1)\|a\|_E^p}{\pi}\int_0^\infty\int_{-\infty}^\infty\frac{1}{[(x-t)^2+(y+u)^2]^{\frac{(\lambda+2)p}{2}-1}}\ \frac{u^\lambda}{y+u}\ \frac{y+u}{(x-t)^2+(y+u)^2}dt\, du$$
$$\le \frac{2^\lambda(\lambda+1)\|a\|_E^p}{\pi}\int_0^\infty\int_{-\infty}^\infty\frac{1}{(y+u)^{(\lambda+2)p-1}}\ u^\lambda\ \frac{y+u}{(x-t)^2+(y+u)^2}dt\, du$$
$$\le 2^\lambda(\lambda+1)\|a\|_E^p\int_0^\infty\frac{u^\lambda}{(y+u)^{(\lambda+2)p-1}}\left(\frac{1}{\pi}\int_{-\infty}^\infty\frac{y+u}{(x-t)^2+(y+u)^2}dt\right)du$$
$$= 2^\lambda(\lambda+1)\|a\|_E^p\int_0^\infty\frac{u^\lambda}{(y+u)^{(\lambda+2)p-1}}\ du$$
$$= 2^\lambda(\lambda+1)\|a\|_E^p\int_y^\infty\frac{1}{u^{(\lambda+2)(p-1)+1}}du \ < \infty,$$

if $p > 1$. Also, since $\omega \longrightarrow K_\lambda^a(z,\omega)$ is analytic, we have $K_\lambda^a(z,.) \in {}_{E,\lambda}(\Lambda^+)$ for $1 < p < \infty$. $\blacksquare$

Proposition 2.6: *For* $1 < p < \infty$, $-1 < \lambda < \infty$, $z = (x,y) \in \Lambda^+$, *there is a constant* $C_{p,\lambda}$

$$\int_{\Lambda^+}u^{-1/p}\ \|K_\lambda^a(z,\omega)\|_E d\mu_\lambda(\omega) \le C_{p,\lambda}\|a\|_E y^{-1/p}$$

Proof:

$$\int_{\Lambda^+}u^{-1/p}\ \|K_\lambda^a(z,\omega)\|_E d\mu_\lambda(\omega)$$

$$= \|a\|_E \int_{\Lambda^+} u^{-1/p} \, \frac{1}{|z-\overline{\omega}|^{\lambda+2}} d\mu_\lambda(\omega)$$
$$= \frac{2^\lambda(\lambda+1)\|a\|_E}{\pi} \int_0^\infty \int_{-\infty}^\infty \frac{u^{\lambda-1/p}}{[(x-t)^2+(y+u)^2]^{\frac{\lambda+2}{2}}} dtdu \qquad (\omega=(t,u))$$
$$\leq 2^\lambda(\lambda+1)\|a\|_E \int_0^\infty \frac{u^{\lambda-1/p}}{[(y+u)^2]^{\frac{\lambda}{2}}} \frac{1}{y+u} du$$
$$\leq 2^\lambda(\lambda+1)\|a\|_E \int_0^\infty \frac{u^{\lambda-1/p}}{(y+u)^{\lambda+1}} du$$
$$\leq 2^\lambda(\lambda+1)\|a\|_E \int_0^\infty \frac{u^{-1/p}}{y+u} du$$
$$= 2^\lambda(\lambda+1)\|a\|_E y^{-1/p} \int_0^\infty \frac{(u/y)^{-1/p}}{y(1+u/y)} du$$
$$= 2^\lambda(\lambda+1)\|a\|_E \, y^{-1/p} \int_0^\infty \frac{v^{-1/p}}{1+v} dv, \qquad \left(\frac{u}{y}=v\right)$$

Since
$$\int_0^\infty \frac{v^{-1/p}}{1+v} dv < \infty,$$
it follows that
$$2^\lambda(\lambda+1) \int_0^\infty \frac{v^{-1/p}}{1+v} dv$$
is constant depending upon λ and p and hence
$$\int_{\Lambda^+} u^{-1/p} \, \|K_\lambda^a(z,\omega)\|_E d\mu_\lambda(\omega) \leq C_{p,\lambda}\|a\|_E y^{-1/p},$$
for some constant $C_{p,\lambda}$. ■

Proposition 2.7 : *Let* $1<p<\infty$, $-1<\lambda<\infty$ *and* $z=(x,y)\in\Lambda^+$. *Then*
$$|||K_\lambda^a(z,.)|||_{p,\lambda}^p \approx \frac{1}{y^{(p-1)(\lambda+2)}}.$$

Proof: We have,
$$|||K_\lambda^a(z,.)|||_{p,\lambda}^p = \int_{\Lambda^+} \|K_\lambda^a(z,\omega)\|_E^p \, d\mu_\lambda(\omega)$$
$$\geq \int_{S(z,ry)} \|K_\lambda^a(z,\omega)\|_E^p \, d\mu_\lambda(\omega)$$
$$\geq C \, \frac{2^\lambda(\lambda+1)\|a\|_E^p}{\pi} \int_{S(z,ry)} \left(\frac{1}{y^{\lambda+2}}\right)^p \, u^\lambda d\vartheta(\omega)$$

$$\geq \ C_1 \frac{2^\lambda(\lambda+1)\|a\|_E^p}{\pi} \frac{1}{y^{p(\lambda+2)-\lambda}} \vartheta\big(S(z,ry)\big)$$

$$= \ C_1 \frac{r^2 2^\lambda(\lambda+1)\|a\|_E^p}{\pi} \frac{1}{y^{(p-1)(\lambda+2)}}$$

$$= \ C_2 \frac{1}{y^{(p-1)(\lambda+2)}}, \text{ where } \quad C_2 = \frac{r^2 2^\lambda(\lambda+1)\|a\|_E^p}{\pi},$$

for $z = (x, y)$ and $\omega = (t, u)$ in Λ^+. We have again,

$$|||K_\lambda^a(z,.)|||_{p,\lambda}^p = \frac{2^\lambda(\lambda+1)\|a\|_E^p}{\pi} \int_0^\infty \int_{-\infty}^\infty \frac{u^\lambda}{[(x-t)^2+(y+u)^2]^{\frac{p(\lambda+2)}{2}}} dt du$$

$$\leq \ 2^\lambda(\lambda+1)\|a\|_E^p \int_0^\infty \frac{1}{(y+u)^{p(\lambda+2)-2}} \frac{u^\lambda}{y+u} du$$

$$\leq \ 2^\lambda(\lambda+1)\|a\|_E^p \int_0^\infty \frac{1}{(y+u)^{(p-1)(\lambda+2)+1}} du$$

$$= \ 2^\lambda(\lambda+1)\|a\|_E^p \int_y^\infty \frac{1}{u^{(p-1)(\lambda+2)+1}} du$$

$$= \ C_3 \frac{1}{y^{(p-1)(\lambda+2)}}, \text{where } \ C_3 = \frac{2^\lambda(\lambda+1)\|a\|_E^p}{(p-1)(\lambda+1)}.$$

Thus,

$$|||K_\lambda^a(z,.)|||_{p,\lambda}^p \approx \frac{1}{y^{(p-1)(\lambda+2)}}.$$

■

3. Carleson-type Theorem

In this section, we establish Carleson-type theorem for weighted vector-valued Bergman spaces of the upper half-plane. This theorem will be used as an effective tool to characterize boundedness of weighted composition operators on these spaces.

Theorem 3.1: Let $0 < r < 1$, $1 \leq p < \infty$ and μ be a positive Borel measure on Λ^+. Then the following are equivalent:

(i) There is a constant $C < \infty$ so that for all $z = (x,y) \in \Lambda^+$, we have $\mu\big(S(z,ry)\big) \leq C\, y^{\lambda+2}$.

(ii) There is a constant $C_1 < \infty$ such that

$$\int_{\Lambda^+} \|f(\omega)\|_E^p \, d\mu(\omega) \ \leq \ C_1 \int_{\Lambda^+} \|f(\omega)\|_E^p \, d\mu_\lambda(\omega),$$

for every $f \in \mathcal{A}_{E,\lambda}^p(\Lambda^+)$.

(iii) There is a constant $C_2 < \infty$ such that

$$\int_{\Lambda^+} \left|\frac{1}{(z-\overline{\omega})^2}\right|^{\lambda+2} d\mu \;\le\; C_2\,\frac{1}{y^{\lambda+2}}.$$

Proof: **(i) ⟹ (ii).** Let $\{z_n\}_{n\in\mathbb{N}}$ be a sequence as choosen in the Lemma 3.1 in [18]. Then

$$\begin{aligned}
\int_{\Lambda^+}\|f(\omega)\|_E^p\, d\mu(\omega) \;&\le\; \sum_{n=1}^{\infty}\int_{S(z_n,ry_n)}\|f(\omega)\|_E^p\, d\mu(\omega)\\
&\le\; \sum_{n=1}^{\infty} sup\{\|f(\omega)\|_E^p :\ \omega\in S(z_n\,,ry_n)\}\,\mu\big(S(z_n,ry_n)\big)\\
&\le\; C'\sum_{n=1}^{\infty}\frac{\mu\big(S(z_n\,,ry_n)\big)}{{y_n}^{\lambda+2}}\int_{S(z_n,3\,ry_n)}\|f(\omega)\|_E^p\, d\mu_\lambda(\omega)\\
&\le\; C\,C'\sum_{n=1}^{\infty}\int_{S(z_n,3\,ry_n)}\|f(\omega)\|_E^p\, d\mu_\lambda(\omega)\\
&\le\; C\,C'M\int_{\Lambda^+}\|f(\omega)\|_E^p\, d\mu_\lambda(\omega)\\
&= C_1\int_{\Lambda^+}\|f(\omega)\|_E^p\, d\mu_\lambda(\omega).
\end{aligned}$$

(ii) ⟹ (iii). Let

$$f(\omega) \;=\; a\left(\frac{i}{z-\overline{\omega}}\right)^{\frac{2(\lambda+2)}{p}}.$$

Then $f \in \mathcal{A}^p_{E,\lambda}(\Lambda^+)$ and

$$\|f(\omega)\|_E^p \;=\; \|a\|_E^p\left|\frac{1}{z-\overline{\omega}}\right|^{2(\lambda+2)}.$$

Therefore, for $z = (x\,,y)$ and $\omega = (t\,,u)$ in Λ^+, we have

$$\begin{aligned}
&\int_{\Lambda^+}\left|\frac{1}{z-\overline{\omega}}\right|^{2(\lambda+2)} d\mu(\omega)\\
&\quad\le \frac{C_1}{\|a\|_E^p}\int_{\Lambda^+}\|f(\omega)\|_E^p\, d\mu_\lambda(\omega)\\
&\quad= \frac{2^\lambda(\lambda+1)}{\pi}C_1\int_0^\infty\int_{-\infty}^{\infty}\frac{1}{[(x-t)^2+(y+u)^2]^{(\lambda+2)}}\,u^\lambda dt du\\
&\quad\le\; 2^\lambda(\lambda+1)C_1\int_0^\infty\frac{1}{(y+u)^{2\lambda+2+1}}\left[\frac{1}{\pi}\int_{-\infty}^{\infty}\frac{y+u}{(x-t)^2+(y+u)^2}dt\right]u^\lambda du\\
&\quad\le\; 2^\lambda(\lambda+1)C_1\int_y^\infty\frac{1}{u^{\lambda+3}}du\\
&\quad=\; C_2\,\frac{1}{y^{\lambda+2}},\qquad \text{where}\;\; C_2 \;=\; 2^\lambda(\lambda+1)C_1.
\end{aligned}$$

(iii) ⟹ (i) Suppose that (iii) holds. Then be Lemma 2.4, we have

$$\int_{\Lambda^+} \|K_\lambda^a(z,\omega)\|_E^2 \, d\mu(\omega) \geq \int_{S(z,\ ry)} \|K_\lambda^a(z,\omega)\|_E^2 \, d\mu(\omega)$$
$$\geq \frac{C\,\|a\|_E^2}{[y^{\lambda+2}]^2}\ \mu\big(S(z,ry)\big) \quad ,$$

for some constant $C > 0$. Thus,

$$\mu\big(S(z,ry)\big) \leq \frac{C'\big(y^{\lambda+2}\big)^2}{\|a\|_E^2} \int_{\Lambda^+} \|K_\lambda^a(z,\omega)\|_E^2 \, d\mu(\omega) \qquad \left[\ C' = \frac{1}{C}\ \right]$$
$$\leq C' \left[y^{\lambda+2}\right]^2 \int_{\Lambda^+} \left|\frac{1}{z-\bar{\omega}}\right|^{2(\lambda+2)} d\mu(\omega)$$
$$\leq C' C_2 \left[y^{\lambda+2}\right]^2 \frac{1}{y^{\lambda+2}}$$
$$\leq C y^{\lambda+2}, \quad \text{where}$$

$C = C' C_2$ ■

A positive Borel measure μ is called a $\lambda -$ Carleson measure if it satisfies any one of the three equivalent conditions in Theorem 3.1.

4. Weighted composition operators on $\mathcal{A}_{E,\lambda}^{p}(\Lambda^+)$

This section is devoted to characterize boundedness of weighted composition operators on weighted Bergman spaces of the upper half-plane.

Theorem 4.1. *Suppose that* $1 \leq p < \infty$, $-1 < \lambda < \infty$, *and let* φ *and* $\psi\colon \Lambda^+ \to \mathbb{C}$ *be analytic maps on* Λ^+ *such that* $\varphi(\Lambda^+) \subset \Lambda^+$.

Then the following are equivalent :

(i) ψC_φ *is a bounded on* $\mathcal{A}_{E,\lambda}^{p}(\Lambda^+)$

(ii) *The pull – back measure* $\vartheta_{\psi,\lambda,p} = \mu_{\psi,\lambda,p}\ o\ \varphi^{-1} of\ \mu_{\psi,\lambda,p}$ *induced by* φ *is a* λ *−Carleson measure. Here* $d\mu_{\psi,\lambda,p} = |\psi|^p d\mu_\lambda$.

(iii)
$$sup\left\{ \int_{\Lambda^+} |\psi(\omega)|^p \frac{y^{\lambda+2}}{|\varphi(\omega)-\bar{z}|^{2(\lambda+2)}} d\mu_\lambda(\omega) \quad : \quad z \in \Lambda^+ \right\} < \infty$$

Proof : (i) ⟺ (ii). By definition ψC_φ is a bounded on $\mathcal{A}_{E,\lambda}^{p}(\Lambda^+)$ if and only if there is a positive constant C_p such that for any $f \in \mathcal{A}_{E,\lambda}^{p}(\Lambda^+)$,

$$\big|\big|\big|(\psi C_\varphi)(f)\big|\big|\big|_{p,\lambda}^p \leq C_p |||f|||_{p,\lambda}^p,$$

which is same as

$$\int_{\Lambda^+} |\psi(z)|^p \,\|f(\varphi(z))\|_E^p \, d\mu_\lambda(z)$$
$$\leq C_p |||f|||_{p,\lambda}^p. \tag{4.1}$$

Let $d\mu_{\psi,\lambda,p}(z) = |\psi(z)|^p d\mu_\lambda(z)$ and let $\vartheta_{\psi,\lambda,p} = \mu_{\psi,\lambda,p}\, o\, \varphi^{-1}$ be the pull – back measure of $\mu_{\psi,\lambda,p}$ induced by φ. If we change variable $\omega = \varphi(z)$, then we get,

$$\int_{\Lambda^+} |\psi(z)|^p \, \|f(\varphi(z))\|_E^p \, d\mu_\lambda(z) = \int_{\Lambda^+} \|f(\varphi(z))\|_E^p \, d\mu_{\psi,\lambda,p}(z) = \int_{\Lambda^+} \|f(\omega)\|_E^p \, d\vartheta_{\psi,\lambda,p}(\omega)$$

Thus (4.1) is equivalent to

$$\int_{\Lambda^+} \|f(\omega)\|_E^p \, d\vartheta_{\psi,\lambda,p}(\omega) \leq C_p \||f\||_{p,\lambda}^p .$$

That is, $\vartheta_{\psi,\lambda,p} = \mu_{\psi,\lambda,p}\, o\, \varphi^{-1}$ is a λ −Carleson measure on Λ^+. Hence (i) and (ii) are equivalent.

(ii) ⟺ (iii). By Theorem 3.1, $\vartheta_\psi = \mu_{\psi,\lambda,p}\, o\, \varphi^{-1}$ is a λ −Carleson measure is equivalent to

$$sup\left\{ \int_{\Lambda^+} \frac{y^{\lambda+2}}{|\omega - \bar{z}|^{2(\lambda+2)}} d\vartheta_{\psi,\lambda,p}(\omega) \quad : \ z \in \Lambda^+ \right\} < \infty.$$

Changing the variable, we get

$$sup\left\{ \int_{\Lambda^+} |\psi(\omega)|^p \frac{y^{\lambda+2}}{|\varphi(\omega) - \bar{z}|^{2(\lambda+2)}} d\mu_\lambda(\omega) \quad : \ z \in \Lambda^+ \right\} < \infty,$$

Thus (ii) and (iii) are equivalent.
□

Corollary 4.2: *Let* $1 \le p < \infty$ *and* $-1 < \lambda < \infty$ *and* φ *be an analytic self map of* Λ^+ *. Then boundedness of the composition operator* C_φ *on* $\mathcal{A}_{E,\lambda}^p(\Lambda^+)$ *, induced by* φ *, is independent of the index p; that is,* φ *simultaneously induces bounded composition operators on all* $\mathcal{A}_{E,\lambda}^p(\Lambda^+)$.

Corollary 4.3. *Let* $1 \le p < \infty$ *and* $-1 < \lambda < \infty$ *and* φ *be a bounded analytic self map of* Λ^+ *and* $\psi: \Lambda^+ \to \mathbb{C}$ *be an analytic map on* Λ^+ *such that* $\psi \notin \mathcal{A}_\lambda^p(\Lambda^+)$. *Then the operators* ψC_φ *,induced by* φ *and* ψ*, is not bounded on* $\mathcal{A}_{E,\lambda}^p(\Lambda^+)$.

Proof: Fix $0 < r < 1$. Choose $z_k = x_k + i\, y_k \in \Lambda^+$, $k = 1,2,3 \ldots, n$ such that

$$\varphi(\Lambda^+) \subset \bigcup_{k=1}^{n} S(z_k, ry_k)$$

for some fixed $r \in (0, 1)$, which is possible because φ is bounded.
Now ,

$$\bigcup_{k=1}^{n} \varphi^{-1}(S(z_k, ry_k) = \varphi^{-1}\left(\bigcup_{k=1}^{n} S(z_k, ry_k)\right) = \Lambda^+.$$

Thus,

$$\mu_{\psi,\lambda,p}\left(\bigcup_{k=1}^{n} \varphi^{-1}(S(z_k, ry_k)\right) = \mu_{\psi,\lambda,p}(\Lambda^+)$$

$$= \int_{\Lambda^+} d\mu_{\psi,\lambda,p}$$
$$= \int_{\Lambda^+} |\psi(z)|^p d\mu_\lambda(z)$$
$$= \infty.$$

That is,

$$\mu_{\psi,\lambda,p}\left(\varphi^{-1}\big(S(z_k, ry_k)\big)\right) = \infty,$$

for some k. Hence C_φ is not bounded. □

As an application of the above result we have

Corollary 4.4. *Let* $1 \le p < \infty$ *and* $-1 < \lambda < \infty$ *and* φ *be a bounded , analytic self mapping of* Λ^+. *Then* φ *does not induce a bounded composition operator on any of the spaces* $\mathcal{A}^p_{E,\lambda}(\Lambda^+)$.

Proof: Since $1 \notin \mathcal{A}^p_\lambda(\Lambda^+)$ for $1 \le p < \infty$ and $-1 < \lambda < \infty$, the proof follows from the above result by taking $\psi \equiv 1$. □

Corollary 4.5. *Let* $1 \le p < \infty$ *and* $-1 < \lambda < \infty$ *and let*

$$\varphi(z) = \frac{az+b}{cz+d}$$

where a, b, c, d are real numbers and ad - bc > 0. Let $\psi: \Lambda^+ \to \mathbb{C}$ *be an analytic map on* Λ^+ *such that* ψC_φ *is bounded on* $\mathcal{A}^p_{E,\lambda}(\Lambda^+)$. *Then either* $c = 0$ *or* $\psi \in \mathcal{A}^p_\lambda(\Lambda^+)$.

Proof: ψC_φ is bounded on $\mathcal{A}^p_{E,\lambda}(\Lambda^+)$. So by corollary 4.3 either $\varphi(z)$ is not bounded or $\psi \in \mathcal{A}^p_\lambda(\Lambda^+)$. Hence either $c = 0$ or $\psi \in \mathcal{A}^p_\lambda(\Lambda^+)$.

□

Corollary 4.6: *Let*

$$\varphi(z) = \frac{az+b}{cz+d},$$

where a, b, c, d are real numbers and ad - bc > 0. Then necessary and sufficient condition for C_φ *to be bounded on* $\mathcal{A}^p_{E,\lambda}(\Lambda^+)$ *is that* $c = 0$.

Corollary 4.7. *Let* φ *and* $\psi: \Lambda^+ \to \mathbb{C}$ *be analytic maps on* Λ^+ *such that* $\varphi(\Lambda^+) \subset \Lambda^+$. *Let the weighted composition operator* ψC_φ *is bounded on the* $\mathcal{A}^p_{E,\lambda}(\Lambda^+)$. *Then*

$$sup\left\{\left(\frac{Im(z)}{Im(\varphi(z))}\right)^{\lambda+2} |\psi(z)|^p : z \in \Lambda^+\right\} < \infty.$$

Proof: Let ψC_φ be bounded on the $\mathcal{A}^p_{E,\lambda}(\Lambda^+)$. Then by Theorem 4.1, we have

$$sup\{B_{\varphi,\lambda}(|\psi|^p)(z) : z \in \Lambda^+\}$$
$$= sup\left\{\int_{\Lambda^+} |\psi(\omega)|^p \frac{y^{\lambda+2}}{|\varphi(\omega) - \bar{z}|^{2(\lambda+2)}} d\mu_\lambda(\omega) : z \in \Lambda^+\right\} < \infty.$$

In particular,

$$sup\left\{\int_{\Lambda^+} |\psi(\omega)|^p \frac{\left(Im\varphi(\alpha)\right)^{\lambda+2}}{\left|\varphi(\omega)-\overline{\varphi(\alpha)}\right|^{2(\lambda+2)}}\, d\mu_\lambda(\omega) \ : z \in \Lambda^+\right\} < \infty.$$

By the subharmonicity of the function

$$\frac{|\psi(\omega)|^p}{\left|\varphi(\omega)-\overline{\varphi(\alpha)}\right|^{2(\lambda+2)}}$$

we get,

$$\int_{\Lambda^+} |\psi(\omega)|^p \frac{\left(Im\varphi(\alpha)\right)^{\lambda+2}}{\left|\varphi(\omega)-\overline{\varphi(\alpha)}\right|^{2(\lambda+2)}}\, d\mu_\lambda(\omega)$$

$$\geq \int_{S(\alpha,\ r\,Im(\alpha))} |\psi(\omega)|^p \frac{\left(Im\varphi(\alpha)\right)^{\lambda+2}}{\left|\varphi(\omega)-\overline{\varphi(\alpha)}\right|^{2(\lambda+2)}}\, d\mu_\lambda(\omega)$$

$$\geq C\left(Im(\alpha)\right)^{\lambda+2} |\psi(\alpha)|^p \frac{\left(Im\varphi(\alpha)\right)^{\lambda+2}}{\left(Im\varphi(\alpha)\right)^{2(\lambda+2)}}$$

$$= C\frac{\left(Im(\alpha)\right)^{\lambda+2}}{\left(Im\varphi(\alpha)\right)^{\lambda+2}} |\psi(\alpha)|^p.$$

□

Corollary 4.8. *Let* $\psi: \Lambda^+ \to \mathbb{C}$ *be an analytic map on* Λ^+ *. Then the operator* M_ψ *is bounded* $\mathcal{A}^p_{E,\lambda}(\Lambda^+)$ *if and only if* $sup\{|\psi(z)| : z \in \Lambda^+\} < \infty$*.*

Corollary 4.9. *Suppose* $\varphi(z) = az + b$*, where* a, b *are real numbers , and* $\psi: \Lambda^+ \to \mathbb{C}$ *be an analytic map on* Λ^+ *such that* M_ψ *is bounded. Then* ψC_φ *is bounded.*

Proof: Using 4.6, we have,

$$\left|\left|\left|\left(\psi C_\varphi\right)(f)\right|\right|\right|^p_{p,\lambda} = \int_{\Lambda^+} |\psi(z)|^p \|f(\varphi(z))\|^p_E\, d\vartheta_\lambda(z)$$

$$\leq M^p \int_{\Lambda^+} \|f(\varphi(z))\|^p_E\, d\vartheta_\lambda(z)$$

$$\leq M^p M_1 \int_{\Lambda^+} \|f(\omega)\|^p_E\, d\vartheta_\lambda(z)$$

$$= M_2 |||f|||^p_{p,\lambda},$$

where

$$M = \sup_{z\in\Lambda^+} |\psi(z)| < \infty, \qquad M_2 = M^p M_1.$$

□

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