Empirical Bayes Inference for the Burr Model Based on Records

Liang Wang and Yimin Shi

Department of Applied Mathematics
Northwestern Polytechnical University
Shaanxi, Xi’an 710072, P.R. China
liang610112@gmail.com
ymshi@nwpu.edu.cn

Abstract

In this paper, the empirical Bayes estimators are derived for the reliability index of Burr type XII model. The Monte-Carlo method are used to investigate the accuracy of the estimators.

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1 Introduction

The two-parameter Burr type XII distribution is a widely used life-time model, which has already gain special attention and intensive study since been introduced by Burr (1942) and are applied extensively in a lot of professions such as reliability, quality control, biostatistics and other fields (see Soliman (2002)). The probability density function (pdf) and cumulative distribution function (cdf) of the Burr type XII distribution denoted by $Burr(c, k)$ are given by

\[
f(x; c, k) = ckx^{c-1}(1+x^c)^{-(k+1)}, F(x; c, k) = 1-(1+x^c)^{-k}, x > 0, c, k > 0 \quad (1)
\]

where $k, c$ are both shape parameters.

For the potential application value in practical situation, a great deal of research has been done on the Burr type XII model. Burr (1942), Rodriguez (1977) introduced the basic statistical property of Burr type XII model. Alimousa et al. (2002) and Lee et al. (2009) obtained the Bayes and empirical Bayes estimators of reliability performances of this model under progressively
type-II censored samples. There are also some authors who pay attentions for its applications and other general Burr model, see for example Giovana (2008).

For the loss function as mentioned below, we will use the symmetric squared error loss and asymmetric Linear-exponential (Linex) loss respectively. In statistical decision theory and Bayes analysis, loss function plays an important role in it, and the squared error loss as the most common symmetric loss function is widely used due to its great analysis property. Under squared loss function, it is to be thought the overestimation and underestimation have the same estimated risks. However, in many practical problems, overestimation and underestimation will make different consequence. Thus using of the symmetric loss function may be inappropriate, and an asymmetric (Linex) loss function has been introduced by Varian (1975), which can be expressed as

\[
L(\delta, \theta) = e^{a(\delta - \phi(\theta))} - a(\delta - \phi(\theta)) - 1, \quad a \neq 0,
\]

which \(\delta\) is any estimate of \(\phi(\theta)\).

From the definition, it can be seen that, when \(a > 0\), the loss function \(L(\delta, \theta)\) is quite asymmetric about zero with overestimation being more costly than underestimation. As \(|\delta - \phi(\theta)| \to \infty\), the loss function \(L(\delta, \theta)\) increase almost exponentially when \(\delta - \phi(\theta) > 0\) and almost linearly when \(\delta - \phi(\theta) < 0\). When \(a < 0\), the linearity-exponentiality phenomenon is reversed. The Bayes estimator of \(\phi(\theta)\) denoted by \(\hat{\phi}_{BL}\) under Linex loss is given by

\[
\hat{\phi}_{BL} = -\frac{1}{a} \ln(\mathbb{E}_\phi[e^{-a\phi(\theta)}]),
\]

Record values was firstly introduced by Chandler (1952), which is a special order statistics from a sample whose size is determined by the values and the order of occurrence of observations. Although the conception of record values was not introduced by a long time as other statistical conceptions like common order statistic, there are a considerable stack of publications on record values.

Let \(X_1, X_2, \cdots\) be a sequence of independent and identically distributed random variable from an arbitrary distribution with cdf \(F(x)\) and pdf \(f(x)\). Set \(Y_k = \max\{X_1, \cdots, X_k\}, k \geq 1\), then \(Y_j\) is an upper record value of the sequence if \(Y_j > Y_{j-1}, j \geq 1\). The sequence of indices at which records happened is defined by for \(L_0 = 1, L_n = \min\{j | j > L_{n-1}, X_j > X_{L_{n-1}}\}, n = 1, \cdots\), then \(\{X_{L_n}, n = 0, 1, \cdots\}\) is called the sequence of upper records of original sequence. Equally, if the sign ‘>’ is changed into ‘<’, we can get the definition of lower records. For more details on record values, see Ahsanullah (1995).

The rest of paper is organized as follows. In section 2, we give the Bayes estimator of the parameter and reliability index under symmetric and asymmetric loss function respectively. In section 3, we obtain the related empirical Bayes estimators by deriving the estimation of the hyper-parameter. In section 4, we provide prediction interval for current records. Finally, section 5 gives a numerical simulation to investigate the accuracy of the estimation.
2 Bayes estimation

In this section, we will consider the Bayes estimation of the Burr type XII model based on record values. For simplicity, let $X_n \equiv X_{L_n}$ be the $n$th upper record value, and suppose $X = (X_1, \ldots, X_n)$ is an upper record values observed from $Burr(c, k)$ with the pdf (1). The likelihood function based on above samples (see Ahsanullah (1995)) is given by

$$L(k, x) = \prod_{i=1}^{n-1} \lambda(x_i) \cdot f(x_n), \quad (4)$$

where $x = (x_1, \ldots, x_n)$ is a sample value of $X$ and $\lambda(\cdot)$ is the failure rate function corresponding to (1) as

$$\lambda(t) = \frac{f(t; c, k)}{R(t)} = \frac{ckt^{c-1}}{1 + ct}, \quad (5)$$

where $R(t) = 1 - F(t; c, k)$ is the survival function at time $t$, given by

$$R(t) = (1 + t^c)^{-k}, t \geq 0. \quad (6)$$

From (1), (4) and (5), the likelihood function $L(k, x)$ is

$$L(k, x) = (1 + x_n^c)^{-k} \cdot \prod_{i=1}^{n} \frac{ckx_i^{c-1}}{1 + x_i^c}, \quad (7)$$

Here we assume that shape parameter $c$ is known, and $k$ is a unknown parameter, which has a gamma conjugate prior pdf $Gam(1, \beta)$ as

$$\pi(k) = \beta e^{-k\beta}, \beta > 0, \quad (8)$$

that is to say, the parameter $k$ be a random variable with exponential distribution $\exp(\beta)$, which is usually used in Bayes statistical inference (see Shi (2005)).

From Bayes theorem, by using (7) and (8), the posterior distribution of $k$, for a given $x$, can be written as

$$\pi(k; x) = \frac{\pi(k) L(k, x)}{\int_{0}^{\infty} \pi(k) L(k, x) dk} = \frac{[\beta + \ln(1 + x_n^c)]^{n+1}}{\Gamma(n+1)} k^n e^{-k[\beta + \ln(1 + x_n^c)]} \quad (9)$$

Now, we start to find the solution of the Bayes estimators of the parameter $k$ and reliability performances $\lambda(t)$ and $R(t)$ under squared loss and Linex loss function respectively.

Under squared error loss function, the Bayes estimators of $k, \lambda(t)$ and $R(t)$, denoted by $k_{BS}, \hat{\lambda}_{BS}$, and $\hat{R}_{BS}$, are the mean of the posterior distribution, as

$$\hat{k}_{BS} = \frac{n + 1}{\beta + \ln(1 + x_n^c)}, \quad \hat{\lambda}_{BS} = \frac{n + 1}{\beta + \ln(1 + x_n^c)} \cdot \frac{ct^{c-1}}{1 + tc}, \quad (10)$$
Furthermore, we have \( \hat{R}_{BS} = \frac{[\beta + \ln(1 + x^c_n)]^{n+1}}{[\beta + 1 + t^c + \ln(1 + x^c_n)]^{n+1}}, \)

Similarly, under Linex loss function, the Bayes estimators of \( k, \lambda(t), \) and \( R(t), \) denoted by \( \hat{k}_{BL}, \hat{\lambda}_{BL}, \) and \( \hat{R}_{BL}, \) can be obtained from (3), as

\[
\hat{k}_{BL} = \frac{n + 1}{a} \ln[\frac{\beta + a + \ln(1 + x^c_n)}{\beta + \ln(1 + x^c_n)}], \hat{\lambda}_{BL} = \frac{n + \frac{c a t^c - 1}{(1 + t^c)(\beta + a + \ln(1 + x^c_n))}}{a}.
\]

and

\[
\hat{R}_{BL} = -\frac{1}{a} \ln \sum_{s=0}^{\infty} \frac{(-a)^s}{s!} + \frac{[\beta + \ln(1 + x^c_n)]^{n+1}}{\beta + s \ln(1 + t^c) + \ln(1 + x^c_n)^{n+1}},
\]

\[\text{(13)}\]

3 Empirical Bayes estimation

In view of the fact that the accuracy of maximum likelihood estimation (MLE) is higher than that of previous estimation such as moment estimation, and that Bayes estimators of \( k, \lambda(t), \) and \( R(t), \) cannot be used directly for the reason of the unknown hyper-parameter \( \beta, \) so in this section, we make use of the maximum likelihood method to estimate \( \beta, \) and then obtain the related empirical Bayes estimators.

As the data are the samples from \( Burr(c, k), \) from (1) and (8), the marginal pdf of \( X \) is given by

\[
f^*(x) = \int_0^\infty \pi(k) \cdot f(x|c, k)dk = \frac{c \beta x^{c-1}}{(1 + x^c)(\beta + \ln(1 + x^c))^2}, x > 0,
\]

\[\text{(14)}\]

and survival function and failure rate function are

\[
R^*(x) = \frac{\beta}{\beta + \ln(1 + x^c)}, \lambda^*(x) = \frac{c \beta x^{c-1}}{(1 + x^c)(\beta + \ln(1 + x^c))}, x > 0,
\]

then the likelihood function of (4) can be expressed as \( L(\beta, x) = \prod_{i=1}^{n} \lambda^*(x_i) f^*(x_n). \) Furthermore, we have \( (\partial/\partial \beta) \ln L(\beta, x) = g_1(\beta) - g_2(\beta), \) where

\[
g_1(\beta) = \frac{1}{\beta} - \frac{1}{\beta + \ln(1 + x^c_n)}, g_2(\beta) = \sum_{i=1}^{n} \frac{1}{\beta + \ln(1 + x^c_i)},
\]

as the MLE of \( \beta \) is needed, we just need to draw a conclusion that the equation \( \frac{\partial \ln L(\beta, x)}{\partial \beta} = 0 \) has only one root. The reason is as follow

\[
g_1'(\beta) = \frac{1}{\beta^2} + \frac{1}{[\beta + \ln(1 + x^c_n)]^2}, g_1''(\beta) = \frac{2}{\beta^3} + \frac{2}{[\beta + \ln(1 + x^c_n)]^3} > 0,
\]
thus, \( g_1(\beta) \) is strict monotone decreasing concave function. Similarly

\[
g_2(\beta) > 0, g_2(\beta) \to 0 (\beta \to \infty), g_2(\beta) \to \sum_{i=1}^{n} \frac{1}{\ln(1 + x_i^\beta)} (\beta \to 0),
\]

\[
g_2'(\beta) = - \sum_{i=1}^{n} \frac{1}{[\beta + \ln(1 + x_i^\beta)]^2} < 0, g_2''(\beta) = \sum_{i=1}^{n} \frac{2}{[\beta + \ln(1 + x_i^\beta)]^3} > 0,
\]

therefore, \( g_2(\beta) \) is also strict monotone decreasing concave function. Furthermore, we also have \( \lim_{\beta \to \infty} \frac{g_1(\beta)}{g_2(\beta)} = 0 \). From above, the equation \( \frac{\partial \ln L(\beta, x)}{\partial \beta} = 0 \) has only one root, and

\[
\beta = [\beta + \ln(1 + x_n^\beta)] + \sum_{i=1}^{n} \frac{1}{[\beta + \ln(1 + x_i^\beta)]} \rightarrow \infty.
\]

Since the exact solution of hyper-parameter \( \beta \) cannot be derived, then an iterative computing method can be obtained to get the solution, the iteration formula is

\[
\beta_{l+1} = \left[ \frac{1}{\beta_l + \ln(1 + x_n^\beta)} + \sum_{i=1}^{n} \frac{1}{\beta_l + \ln(1 + x_i^\beta)} \right]^{-1}, l = 1, 2, \ldots
\]

where \( \beta_l \) is the \( l \)th iterative value, \( \beta_1 \) is an initial value. Suppose the iteration solution is denoted by \( \hat{\beta} \), then the empirical Bayes estimators of \( k, \lambda(t) \), and \( R(t) \), can be obtained by substituting \( \hat{\beta} \) into (10)-(13), donated by \( \hat{k}_{EBS}, \hat{\lambda}_{EBS}, \hat{R}_{EBS}, \hat{k}_{EBL}, \hat{\lambda}_{EBL}, \hat{R}_{EBL} \).

4 Empirical Bayes prediction of future record

In this section, prediction is obtained based on an upper record sample of size \( n \), \( X_1 = x_1, \ldots, X_n = x_n \) are upper record values from \( Burr(c, k) \). Let \( Y \equiv X_s, s > n \) be the \( s \)th upper record value, the conditional probability density function of \( Y \) for given \( X_n = x_n \) is given (see Ahsanullah (1995)) by

\[
h(y | x_n) = \frac{[v(y) - v(x_n)]^{s-n-1}}{\Gamma(s-n)} \cdot \frac{f(y | c, k)}{1 - F(x_n | c, k)}, y > x_n,
\]

where \( v(\cdot) = - \ln(1 - F(\cdot)) \).

For \( Burr(c, k) \) with pdf (1), and the posterior pdf (9), the Bayes prediction function of \( Y \), for given \( x_n \), denoted by \( h^*(y | x_n) \), can be expressed as

\[
h^*(y | x_n) = \int_0^\infty \pi(k; x) \cdot h(y | x_n)dk
\]

\[
= \frac{cy^{c-1}[\beta + \ln(1 + x_n^\beta)]^{n+1}[\ln(1 + y^c) - \ln(1 + x_n^\beta)]^{s-n-1}}{Be(s - n, n + 1)[\beta + \ln(1 + y^c)]^{s+1}}, \quad (17)
\]

where \( Be(u, v) = \frac{\Gamma(u + v)}{\Gamma(u) \Gamma(v)} \) represents the beta function.

Bayes prediction bounds for \( Y \equiv X_s \), given the previous data, are derived by computing \( P(Y > \eta | x_n) \), for some positive \( \eta \). From (17), we have

\[
P(Y > \eta | x_n) = \frac{1}{Be(s - n, n + 1)} G(s - n, n + 1)(\frac{\ln(1 + \eta^c) - \ln(1 + x_n^\beta)}{\beta + \ln(1 + x_n^\beta)}), \quad (18)
\]
where \( G_{u,v}(x) = \int_x^{\infty} \frac{t^{u-1}}{(1+t)^{u+v}} dt \).

In most cases, we will pay close attention to one-step prediction problem. In this special case, \( s = n + 1 \) in (18), it is to be shown that

\[
P(Y > \eta | x_n) = \left[ \frac{\beta + \ln(1 + \eta^c)}{\beta + \ln(1 + x_n^c)} \right]^{-(n+1)},
\]

thus, a 100\(p\)% \( p \in (0, 1) \) Bayesian prediction interval for \( Y = X_{n+1} \) is such that \( P(LL < Y < UL) = p \), where \( LL, UL \) be the lower and upper limits bounds, satisfying

\[
UL = \{\exp[(\beta + \ln(1 + x_n^c))(\frac{1-p}{2})^{\frac{1}{1+p}} - \beta] - 1\}^\frac{1}{c}, \tag{19}
\]

\[
LL = \{\exp[(\beta + \ln(1 + x_n^c))(\frac{1+p}{2})^{\frac{1}{1+p}} - \beta] - 1\}^\frac{1}{c}. \tag{20}
\]

For the same reason as mentioned above, substituting \( \hat{\beta} \) into (19)(20), we obtain the empirical Bayes one-step prediction interval, and the empirical Bayes prediction bounds, donated by \( UL_{EB}, LL_{EB} \).

## 5 Numerical computations

In this section, a numerical example is given to investigate the accuracy of prediction bounds and the empirical Bayes estimators based on Monte-Carlo simulation method.

First, the empirical Bayes one-step prediction bounds for future upper record values are computed according to the following steps.

1. A group of upper record samples from uniform distribution is generated as 0.1805, 0.4305, 0.6649, 0.7865.

2. For given value of \( \beta = 1.06 \) from (8), a group of values of \( k \) are generated according to prior distribution. Take one of them as the value of \( k \), denoted by \( k^* = 0.1938 \).

3. then, for given value of \( c = 3 \) and \( k^* = 0.1938 \), by using inverse translation formula \( x = [(1 - u)^{-\frac{1}{k^*}} - 1]^{\frac{1}{c}} \), where \( u \) stands for above upper record values from uniform distribution, then a group of upper record samples from \( \text{Burr}(c, k) \) are generated as 1.21488, 2.58462, 6.54906, 14.2356.

4. By using the estimate of the hyper-parameter \( \hat{\beta} = 0.8465 \) from (15), 90% empirical Bayes prediction interval for the future upper record value \( Y = X_5 \) is given by (14.673, 15.651).
Empirical Bayes inference for the Burr model

Table 1: Estimated risks (ER) of the EB of \(k, \lambda(t), R(t)\), \((a = 2, t = 10, \beta = 1.06, c = 3)\)

<table>
<thead>
<tr>
<th>size(n)</th>
<th>(k_{EBS})</th>
<th>(k_{EBL})</th>
<th>(\lambda_{EBS})</th>
<th>(\lambda_{EBL})</th>
<th>(\bar{R}_{EBS})</th>
<th>(\bar{R}_{EBL})</th>
</tr>
</thead>
<tbody>
<tr>
<td>n=4</td>
<td>0.02951</td>
<td>0.02167</td>
<td>0.00265</td>
<td>0.00241</td>
<td>0.06871</td>
<td>0.01436</td>
</tr>
<tr>
<td>n=6</td>
<td>0.01791</td>
<td>0.01426</td>
<td>0.00161</td>
<td>0.00150</td>
<td>0.02125</td>
<td>0.01178</td>
</tr>
<tr>
<td>n=9</td>
<td>0.01127</td>
<td>0.00922</td>
<td>0.00101</td>
<td>0.00093</td>
<td>0.01163</td>
<td>0.00822</td>
</tr>
</tbody>
</table>

Table 2: Estimated risks (ER) of the EB of \(k, \lambda(t), R(t)\), \((a = -1, t = 5, \beta = 1.06, c = 2)\)

<table>
<thead>
<tr>
<th>size(n)</th>
<th>(k_{EBS})</th>
<th>(k_{EBL})</th>
<th>(\lambda_{EBS})</th>
<th>(\lambda_{EBL})</th>
<th>(\bar{R}_{EBS})</th>
<th>(\bar{R}_{EBL})</th>
</tr>
</thead>
<tbody>
<tr>
<td>n=4</td>
<td>0.05717</td>
<td>0.06826</td>
<td>0.00846</td>
<td>0.00906</td>
<td>0.27603</td>
<td>0.04183</td>
</tr>
<tr>
<td>n=6</td>
<td>0.01628</td>
<td>0.01829</td>
<td>0.00241</td>
<td>0.00252</td>
<td>0.10461</td>
<td>0.02048</td>
</tr>
<tr>
<td>n=9</td>
<td>0.00802</td>
<td>0.00876</td>
<td>0.00119</td>
<td>0.00123</td>
<td>0.00778</td>
<td>0.01220</td>
</tr>
</tbody>
</table>

Then, the empirical Bayes estimators of \(k, \lambda(t)\) and \(R(t)\), are derived by means of Monte-Carlo simulation. In order to illustrate the accuracy of the different estimators, estimated risk (ER) are computed as the average of their squared deviations. The expression is \(\frac{1}{N} \sum_{i=1}^{N} (\hat{q} - q^*)^2\), where \(N\) denotes times of experiment number by Monte-Carlo simulation method, where \(\hat{q}\) and \(q^*\) denote the original value and empirical estimate of \(k, \lambda(t)\) and \(R(t)\) respectively. Table 1 and Table 2 display the accuracy results about the estimated risks of the estimates of \(k, \lambda(t)\) and \(R(t)\), under different cases of the sample size.

6 Concluding remarks

In this paper, the empirical Bayes estimation and prediction for Burr type XII model based on record values are discussed. There are some conclusions which have been noticed as follows

1. For the good property such as curve-fitting, the Burr type XII model is a widely used model in reliability theory, quality control and many other fields (see Soliman (2002)).

2. It can be shown that, from the form of the estimators, both Bayes and empirical Bayes estimators which are obtained based on the Linex loss function tend to corresponding estimators obtained under squared error loss when \(a\) becomes zero.

3. From Table 1 and Table 2, It may be observed that, the estimated risks of the empirical estimators under both loss function get smaller with increasing samples under different parameter value.
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