A Nonlinear Elasticity Problem
Governed by Lamé System

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Abstract

In this work we study a nonlinear problem equation governed by the system of Lamé, is similar example was the partial differential equations, which operates in relativistic quantum mechanics system. We look for the existence and uniqueness of a function \( u = u(x,t), x \in \Omega, t \in (0,T) \) solution of the problem.

1 Notation

Let \( \Omega \) an open bounded domain of \( IR^n \), with regular boundary \( \Gamma \). We denote by \( u = \{u_1, u_2, u_3, \ldots, u_n\} \) a vector and \( T \) a scalar function on \( Q_\tau = \Omega \times ]0, \tau[ \) where \( \tau \) is a finite real number. \( \lambda, \mu \) are the Lamé coefficients with \( \lambda \geq 0, \mu > 0 \), \( \gamma > 0 \) is a constant. Let \( k > 0 \) be the coefficient of the termic conductivity. Our problem is to study a similar example was the partial differential equations, which operates in relativistic quantum mechanics. It means a \( \Omega \) an open bounded domain of \( \mathbb{R}^n \), with regular boundary \( \Gamma \). We denote by \( Q \) the cylinder \( \mathbb{R}^n_x \times \mathbb{R}_t : Q = \Omega \times (0,T) \), with boundary \( \Sigma \). \( L \) designate Lamé system defined by \( \mu \Delta + (\lambda + \mu) \nabla \text{div} ; \lambda \) and \( \mu \) are constants Lamé with \( \lambda+ \mu \geq 0 \). \( (u_0, u_1, f) \) functions, \( \rho > 0 \). We look for the existence and uniqueness of a function \( u = u(x,t), x \in \Omega, t \in (0,T) \) solution of the problem (1; 2; 3; 4).

\[
\frac{\partial^2 u}{\partial t^2} - Lu + |u|^\rho u = f, \quad x \in \Omega, \ t \in (0,T)
\] (1)

\[
u = 0 \text{ on } \Sigma
\] (2)
\[ u(x, 0) = u_0(x), \quad x \in \Omega \] \hspace{1cm} (3)

\[ \frac{\partial u}{\partial t}(x, 0) = u_1(x), \quad x \in \Omega \] \hspace{1cm} (4)

2 Existence and uniqueness of the solution

2.1 Existence of the solution

The techniques we use are those of the method of compactness

\textbf{Theorem 1} Assume that \( \Omega \) is a bounded open, are given \( f, u_0, u_1 \), with

\[ f \in L^2(Q), \] \hspace{1cm} (5)

\[ u_0 \in H^1_0(\Omega) \cap L^p(\Omega), \quad p = \rho + 2. \] \hspace{1cm} (6)

\[ u_1 \in L^2(\Omega). \] \hspace{1cm} (7)

Then there exists a function \( u \) satisfying:

\[ u \in L^\infty(0, T; H^1_0(\Omega) \cap L^p(\Omega)), \] \hspace{1cm} (8)

\[ \frac{\partial u}{\partial t}(x, 0) \in L^\infty(0, T; L^2(\Omega)), \] \hspace{1cm} (9)

\[ \frac{\partial^2 u}{\partial t^2} - Lu + |u|^{\rho} u = f \text{ in } Q \] \hspace{1cm} (10)

\[ u(0) = u_0(x), \quad x \in \Omega \] \hspace{1cm} (11)

\[ \frac{\partial u}{\partial t}(0) = u_1(x), \quad x \in \Omega \] \hspace{1cm} (12)
2.2 First step: looking for approached solutions

We introduce a sequence functions \( w_1, ..., w_m \) having the following properties:

\[
\begin{cases}
\quad w_i \in H^1_0 (\Omega) \cap L^p (\Omega) \quad \forall i; \\
\quad \forall m, \ w_1, ..., w_m \text{ are linearly independent} \\
\quad \text{combinations of linear finite } w_i \text{ are dense in } H^1_0 (\Omega) \cap L^p (\Omega).
\end{cases}
\]

We look for \( u_m = u_m (t) \) (approximate solution) of the problem as:

\[
u_m (t) = \sum_{i=1}^{i=m} g_{im}(t) w_i.
\]

We determine the functions \( g_{im} \) with the conditions

\[
\left( u''_m (t), w_j \right) + a (u_m, w_j) + (|u_m (t)|^p u_m (t), w_j) = (f (t), w_j), \ 1 \leq j \leq m,
\]

As \( a \) is a bilinear form defined as follows:

\[
a (u, v) = \lambda \sum_{i=1}^{n} \int_{\Omega} \left( \frac{\partial u}{\partial x_i} \frac{\partial v}{\partial x_i} + (\lambda + \mu) \frac{\partial u}{\partial x_i} \frac{\partial v}{\partial x_j} \right) dx
\]

The system (15) of ordinary differential equations nonlinear be supplemented by initial conditions:

\[
u_m (0) = u_{0m}, \quad u_{0m} = \sum_{i=1}^{i=m} \alpha_{im} w_i \rightarrow u_0 \text{ in } H^1_0 (\Omega) \cap L^p (\Omega),
\]

\[
u'_m (0) = u_{1m}, \quad u_{1m} = \sum_{i=1}^{i=m} \beta_{im} w_i \rightarrow u_1 \text{ in } L^2 (\Omega).
\]

Through the linear independence of \( w_1, ..., w_m \), we have \( \det (w_i, w_j) \neq 0 \), ie the system composed of (15), (17) and (18) admits a solution defined on \([0, t_m]\). The a priori estimates which follow show that \( t_m = T \).

2.3 Second step: a priori estimates.

We multiply equation (15) index \( j \) by \( g'_{jm} (t) \) was:

\[
\left( u''_m (t), u'_m (t) \right) + a (u_m (t), u'_m (t)) + (|u_m (t)|^p u_m (t), u'_m (t))
\]
\begin{align*}
= \left( f(t), u_m'(t) \right) \quad (19)
\end{align*}

But, \( |u_m(t)|^p u_m(t) \in L^p(\Omega) \) and \( p = \rho + 2 \), Then according to Cauchy
Schwarz were:
\begin{align*}
\frac{1}{2} \frac{d}{dt} \left( \left| u_m'(t) \right|^2 + \|u_m(t)\|^2 \right) + \frac{1}{p} \frac{d}{dt} \left( \int}\|u_m(x,t)\|^p dx \right) \leq |f(t)| \left| u_m'(t) \right| \quad (20)
\end{align*}

So we integrate between 0, \( t \), we deduce:
\begin{align*}
\frac{1}{2} \left( \left| u_m'(t) \right|^2 + \|u_m(t)\|^2 \right) + \frac{1}{p} \|u_m(x,t)\|_{L^p(\Omega)}^p \leq \frac{1}{2} \|u_{1m}\|^2 + \frac{1}{2} \|u_{0m}\|^2
\end{align*}

\begin{align*}
+ \frac{1}{p} \|u_m(0)\|_{L^p(\Omega)}^p + \int_0^t |f(\sigma)| \left| u_m'(\sigma) \right| d\sigma \quad (21)
\end{align*}

From (17), (18) and inequality:
\begin{align*}
ab \leq \frac{1}{2} a^2 + \frac{1}{2} b^2.
\end{align*}

We have
\begin{align*}
|f(\sigma)| \left| u_m'(\sigma) \right| \leq \frac{1}{2} |f(\sigma)|^2 + \frac{1}{2} \left| u_m'(\sigma) \right|^2.
\end{align*}

Then
\begin{align*}
\frac{1}{2} \left( \left| u_m'(t) \right|^2 + \|u_m(t)\|^2 \right) + \frac{1}{p} \|u_m(x,t)\|_{L^p(\Omega)}^p \leq c + \frac{1}{2} \int_0^t |f(\sigma)|^2 d\sigma
\end{align*}

\begin{align*}
+ \frac{1}{2} \int_0^t \left| u_m'(\sigma) \right|^2 d\sigma. \quad (22)
\end{align*}

But
\begin{align*}
f \in L^2(Q) \Rightarrow \int_0^t |f(\sigma)|^2 d\sigma \leq \text{constant}.
\end{align*}
We deduce, therefore, in particular (22) that:

$$\left| u'_m(t) \right|^2 \leq c' + \int_0^t \left| u'_m(\sigma) \right|^2 d\sigma,$$

(23)

And after the Gronwall inequality we have:

$$\left| u'_m(t) \right| \leq \text{constant}. \quad (24)$$

Then

$$\|u_m(t)\| + \|u_m(x)\|_{L^p(\Omega)} \leq \text{constant}. \quad (25)$$

By (24), (25) and when \(m \to \infty\) we have: \(u_m\) in a bounded set of \(L^\infty(0, T; H^1_0(\Omega) \cap L^p(\Omega))\) and \(u'_m\) in a bounded set of \(L^\infty(0, T; L^2(\Omega))\).

### 2.4 Third step: passage to the limit

In the second step we were \(u_m\) borned in \(L^\infty(0, T; H^1_0(\Omega) \cap L^p(\Omega))\), then it is bounded in \(L^2(0, T; H^1_0(\Omega))\). Since \(L^\infty(0, T; H^1_0(\Omega) \cap L^p(\Omega))\) is the dual of \(L^1(0, T; H^{-1}(\Omega) + L^p(\Omega))\) (resp. \(L^\infty(0, T; L^2(\Omega))\)) there exists a result \(u_m, u_\mu\) such that:

$$\forall g \in L^1\left(0, T; H^{-1}(\Omega) + L^p(\Omega)\right):$$

$$\int_0^T (u_\mu(t), g(t)) \, dt \to \int_0^T (u(t), g(t)) \, dt$$

Which implies:

$$u_\mu \to u \quad \text{weak in } L^\infty(0, T; H^1_0(\Omega) \cap L^p(\Omega)) \quad \text{and in } L^2(0, T; H^1_0(\Omega)). \quad (26)$$

So:

$$\exists u'_\mu \to u' \quad \text{in } D'(0, T; H^1_0(\Omega) \cap L^p(\Omega)) \Rightarrow u'_\mu \to u' \quad \text{in } L^\infty(0, T; L^2(\Omega))$$

and in \(L^2(0, T; L^2(\Omega))\) \quad (27)
Then in particular $u_m$ bounded in $H^1(Q)$, but we know that the next injection is compact:

$$H^1(Q) \hookrightarrow L^2(Q) \quad (28)$$

And according to the definition of compact injection, we can suppose the sequence $u_\mu$ extracted $u_m$ satisfies (26) and (27), then $u, u'$ exists and in $L^2(Q)$ then:

$$\begin{cases} 
    u_\mu \rightarrow u \text{ in } L^2\left(0, T; H^1_0(\Omega) \cap L^p(\Omega)\right) \text{ strong (a.e),} \\
    u'_\mu \rightarrow u' \text{ in } L^2\left(0, T; L^2(\Omega)\right) \text{ weak (a.e).} 
\end{cases} \quad (29)$$

Studying the convergence of $|u_m|^\rho u_m$:

$|u_m|^\rho u_m$ is bounded in $L^\infty\left(0, T; L^p'(\Omega)\right)$, then

We set:

$$|u_\mu|^\rho u_\mu \rightarrow w \text{ in } L^\infty\left(0, T; L^p'(\Omega)\right), \quad (30)$$

Showing that:

$$w = |u|^\rho u. \quad (31)$$

For this we give the following lemma:

**2.4.1 Lemma:**

Let $O$ be an open bounded $\mathbb{R}^n_x \times \mathbb{R}_t$, $g_\mu$ and $g$ des functions of $L^q(O)$, $1 < q < \infty$, such that

$$\|g_\mu\|_{L^q(O)} \leq c, \quad g_\mu \rightarrow g \text{ p.p. in } O$$

Then

$$g_\mu \rightarrow g \text{ in } L^q \text{ weak.}$$

When we ask: $O = Q$ and $g_\mu = |u_\mu|^\rho u_\mu$, from (29):

$$u_\mu \rightarrow u \text{ in } L^2(Q) \text{ (a.e)}$$

Therefore:

$$g_\mu = |u_\mu|^\rho u_\mu \rightarrow |u|^\rho u = g \text{ (a.e) in } L^{p'}(\Omega)$$
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Such that \( p' = \frac{p + 2}{p + 1} = q \) (for \( p = \rho + 2 \)), and after (30):

\[
|u_\mu|^\rho u_\mu \rightharpoonup w \text{ in } L^p(\Omega).
\] (32)

Since the limit is unique, therefore:

\[ g = |u|^\rho u = w. \]

We show that this solution satisfies the equation (15), so when we set \( m = \mu \) and we fix \( j \) such that \( \mu > j \); then:

\[ (u''(t), w_j) + a(u_\mu, w_j) + (|u_\mu|^\rho u_\mu(t), w_j) = (f(t), w_j). \] (33)

From (30) and (31)

\[
(u'_\mu, w_j) \rightharpoonup (u', w_j) \text{ in } L^\infty(0, T) \Rightarrow \frac{d}{dt}(u'_\mu, w_j),
\]

\[
= (u''_\mu, w_j) \rightharpoonup (u'', w_j) \text{ in } D'(0, T)
\] (34)

Where:

\[ a(u_\mu, w_j) \rightharpoonup a(u, w_j) \text{ in } L^\infty(0, T). \]

And after (30) and (31)

\[ (|u_\mu|^\rho u_\mu, w_j) \rightharpoonup (|u|^\rho u, w_j) \text{ in } L^\infty(0, T) \]

It follows therefore from (32) that:

\[
\frac{d^2}{dt^2}(u, w_j) + a(u, w_j) + (|u|^\rho u, w_j) = (f, w_j).
\]

According to the density of the basis \( \{w_j\} \) in separable space \( H_0^1(\Omega) \cap L^p(\Omega) \), we have

\[
\frac{d^2}{dt^2}(u, v) + a(u, v) + (|u|^\rho u, v) = (f, v) \forall v \in H_0^1(\Omega) \cap L^p(\Omega)
\] (35)

Then the solution \( u \) satisfies (4), (5) and (6).

It remains to show that the solution \( u \) satisfies the initial conditions (7), (8): \( u(0) = u_0, u'(0) = u_1 \).

By (26) and (27) we have:

\[ u_\mu \rightharpoonup u \text{ in } L^\infty(0, T; H_0^1(\Omega) \cap L^p(\Omega)), \] (36)
\[
\frac{du_\mu}{dt} = u_\mu' \rightarrow u' \quad \text{in } L^\infty \left(0, T; L^2(\Omega)\right) .
\] (37)

So \( u_\mu \) is continuous on \([0, T]\) then continuous on 0 and then:

\[
u_0\mu = u_\mu(0) \rightarrow u(0) = u_0 \quad \text{in } H^1_0(\Omega) \cap L^p(\Omega) ,
\]

whence (7).

And yet

\[
\left( u''_\mu, w_j \right) \rightarrow \left( u'' , w_j \right) \quad \text{in } L^\infty \left(0, T\right) ,
\]

\[
\left( u'_\mu, w_j \right) \rightarrow \left( u' , w_j \right) \quad \text{in } L^\infty \left(0, T\right) .
\]

Then

\[
\left( u'_\mu(0) , w_j \right) \rightarrow \left( u' , w_j \right) \big|_{t=0} = \left( u'(0) , w_j \right) ,
\]

According (14):

\[
\left( u'_\mu(0) , w_j \right) \rightarrow (u_1, w_j) .
\]

We have:

\[
\left( u' (0) , w_j \right) = (u_1, w_j) , \forall j.
\]

Then:

\[
u' (0) = u_1 ,
\]

Whence (8).

3 Uniqueness of solution

3.0.2 Theorem 2 :

It is located in the assumptions of Theorem 1 with:

\[
\rho \leq \frac{2}{n - 2}
\] (38)

(any finite \( \rho \) if \( n = 2 \)). Then the solution \( u \) obtained in Theorem one is unique.
3.0.3 Preuve :

Let $u, v$ be two solutions, in the sense of Theorem 2, then $w = u - v$ satisfies:

$$
\begin{aligned}
\begin{cases}
  w'' - Lw = |v|^\rho v - |u|^\rho u, \\
  w(0) = 0, \quad w'(0) = 0, \\
  w \in L^\infty (0, T; H^1_0 (\Omega) \cap L^p (\Omega)), \\
  w' \in L^\infty (0, T; L^2 (\Omega)).
\end{cases}
\end{aligned}
$$

So (28) implies:

$$(w'', v) + a (w, v) = (|v|^\rho v - |u|^\rho u , v) \quad \forall v \in H^1_0 (\Omega).$$

To replace $v$ by $w'$ must $w' \in H^1_0 (\Omega)$ for $v \in H^1_0 (\Omega)$ but $w' \in L^2 (\Omega)$ then we must introduce an auxiliary function:

$$
\forall s \in \mathbb{R}, T[
\begin{array}{c}
\Psi : [0, T[ \rightarrow \mathbb{R} \\

s \mapsto \Psi (t) = \begin{cases}
  -\int_s^t w (\sigma) d\sigma, \ t \leq s; \\
  0, \quad t > s
\end{cases}
\end{array}
$$

$$
\Psi' (t) = w (t); \quad w_1 (t) = \int_0^t w (\sigma) d\sigma \text{ if } \forall t \leq s.
$$

Thus

$$
\Psi (t) = -\int_t^s w (\sigma) d\sigma = w_1 (t) - w_1 (s) \Rightarrow \Psi (0) = -w_1 (s)
$$

Then (28) gives:

$$
(w'', \Psi (t)) + a (w, \Psi (t)) = (|v|^\rho v - |u|^\rho u , \Psi (t)) \Rightarrow \\
\frac{1}{2} |w (s)|^2 + \frac{1}{2} \|w_1 (s)\|^2 = -\int_0^s (|v|^\rho v - |u|^\rho u , \Psi (t)) dt
$$
We have

\[ | - (|v|^\rho v - |u|^\rho u, \Psi(t))| \leq c \int_\Omega \sup (|u|^\rho, |v|^\rho) |u - v| |\Psi(t)| \, dx = \]

\[ c \int_\Omega \sup (|u|^\rho, |v|^\rho) |w(t)| |\Psi(t)| \, dx \]

According to Holder’s inequality we have:

\[ c \int_\Omega \sup (|u|^\rho, |v|^\rho) |w(t)| |\Psi(t)| \, dx = c \int_\Omega \sup (|u|^\rho, |v|^\rho) |w(t)|. \]

\[ |w_1(t) - w_1(s)| \, dx \leq c \left[ ||u|^\rho||_{L^\rho(\Omega)} + ||v|^\rho||_{L^\rho(\Omega)} \right] \]

As

\[ \frac{1}{n} + \frac{1}{q} + \frac{1}{2} = 1, \]

Then

\[ \frac{1}{q} = \frac{n - 2}{2n} \Rightarrow q = \frac{2n}{n - 2}. \]

But in Theorem 2

\[ \rho \leq \frac{2}{n - 2} \Rightarrow \rho n \leq \frac{2n}{n - 2} = q \Rightarrow \rho n \leq q. \]

We have

\[ H_0^1(\Omega) \subset L^q(\Omega), \quad \frac{1}{q} = \frac{1}{2} - \frac{1}{n}, n \geq 3 \]

Then

\[ \left| \int_\Omega (|u|^\rho u - |v|^\rho v) \Psi(t) \, dx \right| \leq c \left( ||u(t)||^\rho + ||v(t)||^\rho \right) \left( ||w_1(t)||_{L^3(\Omega)} + ||w_1(s)||_{L^3(\Omega)} \right) ||w(t)||_{L^2(\Omega)} \]

And as \( u, v \in L^\infty(0, T; H_0^1(\Omega)) \) for \( H_0^1(\Omega) \subset L^q(\Omega) \), we have:

\[ \left| \int_\Omega (|u|^\rho u - |v|^\rho v) \Psi(t) \, dx \right| \leq c |w(t)| (||w_1(t)|| + ||w_1(s)||). \]
So
\[ |w(s)|^2 + \|w_1(s)\|^2 \leq c \int_0^s \left( |w(t)|^2 + \|w_1(t)\|^2 \right) dt. \]

According to Gronwall inequality we have:
\[ |w(s)|^2 + \|w_1(s)\|^2 = 0 \Rightarrow \begin{cases} w(s) = 0 \\ w_1(s) = 0 \end{cases} \Rightarrow \begin{cases} u = v \\ u' = v' \end{cases}. \]

Then we have the uniqueness.

References


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