1 Introduction

Let $\hat{Q}$ be a bounded open set of $\mathbb{R}^n_x \times (0, T)$, $T > 0$. We define

$$\Omega_s = \hat{Q} \cap \{t = s; 0 \leq s \leq T\}$$

and suppose that the sets $\Omega_s$ are open for all $s$. We represent by $\Gamma_s$ the smooth boundary of $\Omega_s$. The lateral boundary of $\hat{Q}$ is given by

$$\hat{\Sigma} = \bigcup_{0 < s < T} \Gamma_s \times \{s\}$$

The boundary of $\hat{Q}$ is defined by

$$\partial \hat{Q} = \Omega_0 \cup \hat{\Sigma} \cup \Omega_T$$

where, $\Omega_0$ is bounded open set of $\mathbb{R}^n_x$ with $x = (x_1, x_2, \ldots, x_n)$. Let $\Omega$ be a bounded open set of $\mathbb{R}^n_x$ and denote by $Q = \Omega \times (0, T)$ a cylinder such that $\hat{Q} \subset Q$. Let $\Gamma$ be the boundary of $\Omega$ also smooth and let $\Sigma = \Gamma \times (0, T)$ the lateral boundary of the cylinder $Q$. Let $\Omega$ be a bounded open set in $\mathbb{R}^n$ with boundary $\Gamma$ smooth and let $T$ is a positive real number.
In the set $\widehat{Q}$ we will consider the following problem:

$$
\begin{aligned}
|u'| + Au &= f \\
u(0) &= u_0
\end{aligned}
$$

(1)

where, $A$ is the pseudo Laplacian operator.

The problem (1) in cylinder domain was solved in J.L.Lions [2] by Compactness Method. Also in J.L.Lions [2] was given by other solution of this problem utilizing the Monotony Method, due to M.Visik [7].

An problem in manifolds with this operator was study by authors, to appear [5].

In this work we will analyze the problem (1) in the Non Cylindric Domain $\widehat{Q}$.

We will use the Penazation Method, idealized by J.L.Lions and the Monotony Method.

The proof consist in transform the problem (1) in a problem in the cylinder $Q$, solve and then restrict the problem to the non cylinder domain $\widehat{Q}$.

2 Notations, Hypotheses

All derivates are in the distribution sense. By $\mathcal{D}(\Omega)$ we will denote the space of the testes functions in $\Omega$.

We will represent by $W^{1,p}_0(\Omega)$ the closed of $\mathcal{D}(\Omega)$ in $W^{1,p}(\Omega)$. The dual space of $W^{1,p}_0(\Omega)$ is denote by $W^{-1,p'}(\Omega)$, where $p'$ denote the conjugate exponent of $p$, that is, $\frac{1}{p} + \frac{1}{p'} = 1$.

Let $A$ the pseudo Laplacian operator, that is,

$$
A : W^{1,p}_0(\Omega) \to W^{-1,p'}(\Omega)
$$

tal que

$$
A(w) = -\sum_{i=1}^{n} \frac{\partial}{\partial x_i} \left( |\frac{\partial w}{\partial x_i}|^{p-2} \frac{\partial w}{\partial x_i} \right), \; 2 < p < \infty.
$$

We remind that the operator $A$ has the followings proprieties:

- $A$ is bounded, that is, carry bounded in bounded;
- $A$ monotonic, hemicontinuous, $\langle A(u), u \rangle = \|u\|_{W^{1,p}_0}^p$, coercive.

We go assume the following hypotheses:

(H1) The family open $\{\Omega_s\}_{0 < s < T}$ is increasing in the following sense.

If $t_1 \leq t_2$ then $\text{proj}_{\mathbb{R}^n} \Omega_{t_1} \subseteq \text{proj}_{\mathbb{R}^n} \Omega_{t_2}$

(H2) Regularity of the boundary of $\widehat{Q}$

If $v \in W^{1,p}_0(\Omega)$ and $v = 0$ q.s in $\Omega - \Omega_t$ then $v \in W^{1,p}_0(\Omega_t)$. 
Finally, we consider the function
\[ M(x, t) = \begin{cases} 1, & \text{in } Q - \hat{Q} \cup \{\Omega_0 \times \{0\}\} \\ 0, & \text{in } \hat{Q} \cup \Omega_0 \times \{0\} \end{cases} \]
and \( \beta(u) = \frac{1}{\epsilon} M(x, t) u, \forall \epsilon > 0. \)
We note that \( M \in L^\infty(Q). \)

**Definition 2.1** The function \( u : \hat{Q} \to \mathbb{R} \) is a weak solution of the problem (1) if \( u \in L^p(0, T; W^{1, p}_0(\Omega_t)) \) and
\[
\frac{d}{dt}(u(t), v) + \langle Au(t), v \rangle = (f(t), v) \text{ in } D'(\Omega_t),
\]
for all \( v \in W^{1, p}_0(\Omega_t) \)
\( u(0) = u_0 \)

### 3 Main Result

In this section we will solve the follow result

**Theorem 1** Given \( f \in L^p(0, T; W^{-1, p'}(\Omega_t)) \) and \( u_0 \in W^{1, p}_0(\Omega_t) \), then there exists a unique solution of the problem (1) in the sense of the definition 2.1.

The idea of proof consist in transform the problem (1) in a equivalent problem in the cylinder utilizing the penalization method.

#### 3.1 Penalized Problem

Given \( \epsilon > 0 \) to each function \( u_\epsilon : Q \to \mathbb{R} \) solution of the problem:
\[
\begin{align*}
    u_\epsilon' + Au_\epsilon + \frac{1}{\epsilon} Mu_\epsilon &= \tilde{f} \text{ in } Q \\
    u_\epsilon &= 0 \text{ on } \Sigma \\
    u_\epsilon(x, 0) &= \tilde{u}_0 \text{ in } \Omega
\end{align*}
\]
where
\[
\tilde{f}(x, t) = \begin{cases} f(x, t) & \text{in } \hat{Q} \\ 0 & \text{in } Q - \hat{Q} \end{cases}
\]
and
\[
\tilde{u}(x, 0) = \begin{cases} u_0 & \text{in } \Omega_0 \\ 0 & \text{in } \Omega - \Omega_0 \end{cases}
\]
where \( \tilde{u}_0 \in W^{1, p}_0(\Omega) \).
From separability of \( V = W^{1, p}_0(\Omega) \) there exists an hilbetian’s base \((w_\nu)_\nu \subset V. \)
Let \( V_m = [w_1, \ldots, w_m] \) be the subspace of \( V \) generate by \( m \) first vectors of \((w_\nu)_\nu. \)
3.2 Approximated Problem

Consider \( u_{em}(t) \in V_m \) such that:

\[
\begin{align*}
|u_{em}(t) & \in V_m \\
(u'_{em}(t), v) + (Au_{em}(t), v) + \\
\frac{1}{\varepsilon} (Mu_{em}(t), v) & = (\tilde{f}(t), v), \forall v \in V_m \\
\end{align*}
\]

(4)

Hence, the system (4) has a local solution on the interval \([0, t_m)\), with \(t_m < T\). This solution can be extended to the whole interval \([0, T]\) as consequence of the priori estimates that shall be proved in the next step.

3.3 Estimates I

Considering \( v = u_{em}(t) \) in (4) and using the proprieties of the operator \( A \) we have the existence of a subsequence \((u_{\nu}) \subset (u_{em})\) such that:

\[
\begin{align*}
\frac{1}{\varepsilon} (Mu_{\nu}(t), v) & \rightarrow \zeta in L^2(\Omega) \\
u_{\nu} & \rightarrow u_{\varepsilon} in L^\infty(0, T, L^2(\Omega)) \\
u_{\nu} & \rightarrow u_{\varepsilon} in L^p(0, T, W^{1,p}_0(\Omega)) \\
Au_{\nu} & \rightarrow \chi in L^{p'}(0, T, W^{-1,p'}(\Omega)) \\
\end{align*}
\]

(5) (6) (7) (8)

Writing the approximated equation with \( \nu \), multiplying by \( \varphi \in D(0, T) \), integrating from 0 to \( T \) and integrating by parts we obtain:

\[
\begin{align*}
\int_0^T (u_{\nu}(t), \varphi(t))dt + \int_0^T (Au_{\nu}(t), \varphi(t))dt \\
+ \int_0^T \frac{1}{\varepsilon} (Mu_{\nu}(t), \varphi(t))dt & = \int_0^T (\tilde{f}(t), \varphi(t))dt, \\
\forall v \in V_m. \\
\end{align*}
\]

(9)

3.4 Convergence of the term: \( \frac{1}{\varepsilon} (M(t)u_{\nu}(t), v) \)

As \( u_{\nu} \) is bounded in \( L^\infty(0, T; L^2(\Omega)) \hookrightarrow L^2(0, T; L^2(\Omega)) = L^2(Q) \), hence \( u_{\nu} \) is bounded in \( L^2(Q) \). Therefore,

\[
u_{\nu} \rightarrow u_{\varepsilon} in L^2(Q)
\]

(10)

But, \( M\varphi \in L^2(Q) \), because \( M \in L^\infty(Q) \). Therefore \((u_{\nu}, M\phi) \rightarrow (u_{\varepsilon}, M\phi), \forall \phi \in L^2(Q)\).
Taking to the limit in (9) when \( \nu \to \infty \), using the convergence obtained and using the density of \( V_m \) in \( V \) and we have:
\[
\frac{d}{dt}(u_\epsilon(t), v) + (\chi(t), v) + \frac{1}{\epsilon}(M u_\epsilon(t), v) = (\tilde{f}(t), v), \quad \forall \, v \in V, \text{ in the sense of } D'(0, T).
\] (11)

To show that, \( \chi(t) = A(u_\epsilon(t)) \), we used the your monotony and hemicontinuity. While that the verification of \( u_\epsilon(0) = \tilde{u}_0 \) and \( u_{\epsilon m}(T) \to u_\epsilon(T) \) is done form standard.

Thus, by Teman’s Lemma [6] we have
\[
\frac{d}{dt}(u_\epsilon, v) + (A(u_\epsilon), v) + \frac{1}{\epsilon}(M u_\epsilon, v) = (\tilde{f}(t), v), \quad \forall \, v \in V, \text{ in the sense of } D'(0, T).
\] (12)

Multiplying (12) by \( v = u_\epsilon \), we have, as in the estimates I, when \( \epsilon \to 0 \)
\[
u \to \infty, \text{ using the convergence obtained and using the density of } V_m \in V \text{ and we have:}
\[
\frac{d}{dt}(u_\epsilon(t), v) + (\chi(t), v) + \frac{1}{\epsilon}(M u_\epsilon(t), v) = (\tilde{f}(t), v), \quad \forall \, v \in V, \text{ in the sense of } D'(0, T).
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\[
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\] (11)

From estimates, we obtain, when \( \epsilon \to 0 \), \( M u_\epsilon \to 0 \) in \( L^2(0, T, L^2(\Omega)) \), where \( Mw = 0 \) a.s. in \( Q \). Therefore
\[
\frac{d}{dt}(u_\epsilon, v) + (A(u_\epsilon), v) + \frac{1}{\epsilon}(M u_\epsilon, v) = (\tilde{f}(t), v), \quad \forall \, v \in V, \text{ in the sense of } D'(0, T).
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\[
u \to \infty, \text{ using the convergence obtained and using the density of } V_m \in V \text{ and we have:}
\] (11)

De (14) e (16) and of the hypotheses \((H2)\), if \( u \) to design the restriction of \( w \) the \( \hat{Q} \), we have
\[
u \to \infty, \text{ using the convergence obtained and using the density of } V_m \in V \text{ and we have:}
\] (11)

3.5 Restriction the \( \hat{Q} \)

The restriction of the equation (12) to \( \hat{Q} \), is
\[
u \to \infty, \text{ using the convergence obtained and using the density of } V_m \in V \text{ and we have:}
\] (12)

where \( \hat{u}_\epsilon \) represent the restriction of \( u_\epsilon \) a \( \hat{Q} \)

As \( \hat{u}_\epsilon \in C_s([0, T], W_{0}^{1,p}(\Omega_t)) \) we have that the application \( t \to (\hat{u}_\epsilon(t), y) \) is continuous for \( y \in W^{-1,p'}(\Omega_t) \), hence multiplying the equation (17) by \( \theta \in D(0, T) \), integrating from 0 to \( T \) an integrating by parts we obtain
\[
u \to \infty, \text{ using the convergence obtained and using the density of } V_m \in V \text{ and we have:}
\] (12)

where \( \hat{u}_\epsilon \) represent the restriction of \( u_\epsilon \) a \( \hat{Q} \)

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\[
u \to \infty, \text{ using the convergence obtained and using the density of } V_m \in V \text{ and we have:}
\] (12)

where \( \hat{u}_\epsilon \) represent the restriction of \( u_\epsilon \) a \( \hat{Q} \)

As \( \hat{u}_\epsilon \in C_s([0, T], W_{0}^{1,p}(\Omega_t)) \) we have that the application \( t \to (\hat{u}_\epsilon(t), y) \) is continuous for \( y \in W^{-1,p'}(\Omega_t) \), hence multiplying the equation (17) by \( \theta \in D(0, T) \), integrating from 0 to \( T \) an integrating by parts we obtain
\[
u \to \infty, \text{ using the convergence obtained and using the density of } V_m \in V \text{ and we have:}
\] (12)
As \( u, \tilde{u}_\epsilon \) are the restrictions of \( w, u_\epsilon \) respectively, we have of (13) and (14), when \( \epsilon \to 0 \)
\[
\tilde{u}_\epsilon \rightharpoonup u \text{ in } L^\infty(0,T, L^2(\Omega_t))
\]
\[
\hat{u}_\epsilon \to u \text{ in } L^p(0,T, W^1_0(\Omega_t))
\]
\[
A\hat{u}_\epsilon \to \xi \text{ in } L^p(0,T, W^{-1,p'}(\Omega_t))
\]
\[
\hat{u}_\epsilon(T) \to \beta \text{ in } L^2(\Omega_t)
\]

Analogously, as in the first part of the proof, show that \( \beta = u(T) \text{ and } \xi = Au \).

Taking to the limit in (18) when \( \epsilon \to 0 \) and using the convergence obtained we have
\[
\frac{d}{dt}(u(t), v) + (A(u(t)), v) = (f(t), v),
\]
\[\forall v \in W^1_0(\Omega_t) \text{ em } \mathcal{D}'(0,T).\]

As \( u \in C^0([0,T], W^{-1,p'}(\Omega)) \) make sense calculate \( u(0) \).

Being by first part of the proof \( u_\epsilon(0) = \tilde{u}_0 \) we have that \( \hat{u}_\epsilon(0) = u_0 \) where we conclude \( u(0) = u_0 \).

For to show the uniqueness is used the monotony of the pseudo Laplacian operator \( A \). What that conclude the proof of the Theorem 1.

### 3.6 Asymptotic Behavior

The solution from Theorem 1 can be extended the interval \([0, \infty)\), hence we make sense to think in decay.

From (17) with the \( v = \tilde{u}_\epsilon \), the energy of the solution associated to the restrict system (3) to \( \hat{Q} \) is given by \( E_\epsilon(t) = \frac{1}{2}|\tilde{u}_\epsilon|^2 \).

Taking the duality (3)1 restrict to \( \hat{Q} \) com \( \tilde{u}_\epsilon \) we have
\[
\frac{1}{2} \frac{d}{dt}|\tilde{u}_\epsilon|^2 + \|\hat{u}_\epsilon\|^p = 0
\]

where we obtain \( \frac{1}{2} \frac{d}{dt}|\tilde{u}_\epsilon|^2 \leq 0 \), that is, \( \frac{d}{dt}E_\epsilon(t) \leq 0, \forall t \geq 0 \).

Therefore, \( E_\epsilon \) is a nonnegative increasing function.

Integrating (23) de 0 a \( t \) we have \( E_\epsilon(t) + \int_0^t \|\hat{u}_\epsilon\|_V^p = E_\epsilon(0) \). Where, we obtain
\[
E_\epsilon(t) - E_\epsilon(t + 1) = \int_0^t \|\hat{u}_\epsilon(s)\|_V^p ds.
\]

Using the immersion of \( W^1_0(\Omega_t) \) in \( L^2(\Omega_t) \) we obtain
\[
\int_t^{t+1} |\tilde{u}_\epsilon|^2 ds \leq c_1 \int_t^{t+1} \|\hat{u}_\epsilon\|_V^p.
\]
Thus
\[ \int_t^{t+1} |\hat{u}_\epsilon|^2 ds \leq C[E_\epsilon(t) - E_\epsilon(t + 1)] = F^2(t) \] (24)

We consider now the subintervals \((t, t + \frac{1}{4})\) and \((t + \frac{3}{4}, t + 1)\) of \((t, t + 1)\). Using the Medium Value Theorem for integrals, we have that there exists \(t_1 \in (t, t + \frac{1}{4})\) such that
\[ \frac{1}{4} |\hat{u}_\epsilon| = \int_t^{t+\frac{1}{4}} |\hat{u}_\epsilon|^2 ds \leq \int_t^{t+1} |\hat{u}_\epsilon|^2 ds \leq F^2(t) \] (25)

Where, we obtain \(|\hat{u}_\epsilon(t_1)| \leq 2F^2(t)\).

Analogously we obtain \(t_2 \in (t + \frac{3}{4}, t + 1)\) such that \(|\hat{u}_\epsilon(t_2)| \leq 2F^2(t)\).

Integrating the energy in \([t_1, t_2]\) and using the Medium Value Theorem for integrals, we have that there exists \(t^* \in (t_1, t_2)\) such that
\[ (t_2 - t_1)E_\epsilon(t^*) = \int_{t_1}^{t_2} E_\epsilon(s) ds \leq F^2(t) \]

As \(t_2 - t_1 > \frac{1}{2}\) we have that: \(E_\epsilon(t^*) \leq 2F^2(t)\).

Let \(\tau_1, \tau_2 \in [t, t + 1]\) with \(\tau_1 < \tau_2\) and \(\tau_1 = t^*\). We have
\[ E_\epsilon(\tau_2) \leq E_\epsilon(t^*) + \int_t^{t+1} \|\hat{u}_\epsilon\|^p ds, \ \forall \tau_2 \in [t, t + 1] \]

Where, we obtain
\[ \sup_{t \leq s \leq t+1} E_\epsilon(s) \leq E_\epsilon(t^*) + \int_t^{t+1} \|\hat{u}_\epsilon(s)\|^p ds \]

Thus and noting that
\[ \int_t^{t+1} \|\hat{u}_\epsilon\|^p ds \leq \frac{1}{C} F^2(t), \]
we obtain
\[ \sup_{t \leq s \leq t+1} E_\epsilon(s) \leq C[E_\epsilon(t) - E_\epsilon(t + 1)] \]

Therefore, by Nakao’s Lemma [3], we have
\[ E_\epsilon(t) \leq Ce^{-\delta t}, \ \forall \epsilon > 0. \]

We have that \(\hat{u}_\epsilon(t) \to u(t)\) in \(L^2(\Omega_t)\), when \(\epsilon \to 0\). Using this convergence and taking to the inferior limit in the inequality above, when \(\epsilon \to 0\), we obtain:
\[ E(t) \leq Ce^{-\delta t}. \]

What that characterize the asymptotic behavior
References


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