Modified Laplace Decomposition Method

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Abstract

In this paper, a reliable modification of Laplace decomposition method (LDM), namely the modified Laplace decomposition method (MLDM) is presented. The scheme is tested for some examples and the results demonstrate reliability and efficiency of the proposed method.

Keywords: Modified Laplace decomposition method, Exact solution, Nonlinear partial differential equations, Noise terms phenomena

1 Introduction

The Adomian decomposition method (ADM) introduced by Adomian [1 – 2] possesses a great potential in solving different kinds of functional equations. Nonlinear phenomena, that appear in many areas of scientific fields such as solid state physics, plasma physics, fluid mechanics, population models and chemical kinetics can be modelled by nonlinear differential equations. The Adomian decomposition method has been proved to be effective and reliable for handling differential equations, linear or nonlinear. In the nonlinear case for ordinary differential equations and partial differential equations, the method has the advantage of dealing directly with the problem [3 – 8]. These equations are solved without transforming them to more simple ones. The method avoids linearization, perturbation, discretization, or any unrealistic assumptions [9 – 10].

While dealing with nonlinear ordinary differential or partial differential equations, the nonlinearity is replaced by a series of what are called Adomian...
polynomials [11]. The evaluation of these polynomials is necessary, as they contribute to the solution’s series components.

Recently, Adomian and Rach [12] introduced the phenomena of the so-called "noise terms". The "noise terms" were defined in [12] as the identical terms with opposite signs that appear in the components of the series solution of \( u(x) \). In [12], it is concluded that if terms in the component \( u_0 \) are cancelled by terms in the component \( u_1 \), even through \( u_1 \) contains further terms, then the remaining non-cancelled terms of \( u_0 \) provide the exact solution. It was suggested in [12] that the noise terms appears always for inhomogeneous equations. Most recently, Wazwaz [13] established a necessary condition that is essentially needed to ensure the appearance of "noise terms" in the inhomogeneous equations. The necessary condition for the "noise terms" to appear in the components \( u_0 \) and \( u_1 \) is that the exact solution \( u(x) \) must appear as the part of \( u_0 \) among other terms.

In this work, we will use the modified form of Laplace decomposition method introduced by Khuri [14–15]. This powerful modification of Laplace decomposition method will be proposed in section 2. The proposed modification will accelerate the rapid convergence of series solution when compared with Laplace decomposition method and therefore provides major progress. Agadjanov [16] solved Duffing equation with the help of this method. This numerical technique basically illustrates how the Laplace transform may be used to approximate the solutions of the nonlinear partial differential equations by manipulating the decomposition method. Elgasery [17] applied Laplace decomposition method for the solution of Falkner-Skan equation. Here modified Laplace decomposition is implemented to three nonlinear partial differential equations [18]. The effectiveness and the usefulness of modified Laplace decomposition method are demonstrate by finding exact solutions of these three models.

2 Modified Laplace decomposition method

The purpose of this section is to discuss the use of modified Laplace transform algorithm for the nonlinear partial differential equations. For convenience we consider the general form of second order nonhomogeneous nonlinear partial differential equations with initial conditions is given below

\[
Lu(x,t) + Ru(x,t) + Nu(x,t) = h(x,t),
\]

\[
u(x,0) = f(x), \quad u_t(x,0) = g(x),
\]
where $L$ is second order differential operator $L = \frac{\partial^2}{\partial t^2}$, $R$ is the is remaining linear operator, $Nu$ represents a general non-linear differential operator and $h(x, t)$ is source term. The methodology consists of applying Laplace transform first on both sides of Eq. (2.1)

\[ \mathcal{L} \left[ Lu(x, t) \right] + \mathcal{L} \left[ Ru(x, t) \right] + \mathcal{L} \left[ Nu(x, t) \right] = \mathcal{L} \left[ h(x, t) \right]. \]  

(2.3)

Using the differentiation property of Laplace transform we get

\[ s^2 \mathcal{L} \left[ u(x, t) \right] - sf(x) - g(x) + \mathcal{L} \left[ Ru(x, t) \right] + \mathcal{L} \left[ Nu(x, t) \right] = \mathcal{L} \left[ h(x, t) \right], \]  

(2.4)

\[ s^2 \mathcal{L} \left[ u(x, t) \right] - sf(x) - g(x) + \mathcal{L} \left[ Ru(x, t) \right] + \mathcal{L} \left[ Nu(x, t) \right] = \mathcal{L} \left[ h(x, t) \right], \]  

(2.5)

\[ \mathcal{L} \left[ u(x, t) \right] = \frac{f(x)}{s} + \frac{g(x)}{s^2} + \frac{1}{s^2} \mathcal{L} \left[ h(x, t) \right] - \frac{1}{s^2} \mathcal{L} \left[ Ru(x, t) \right] - \frac{1}{s^2} \mathcal{L} \left[ Nu(x, t) \right]. \]  

(2.6)

The second step in Laplace decomposition method is that we represent solution as an infinite series given below

\[ u = \sum_{n=0}^{\infty} u_n(x, t). \]  

(2.7)

The nonlinear operator is decompose as

\[ Nu(x, t) = \sum_{n=0}^{\infty} A_n, \]  

(2.8)

Where $A_n$ are Adomian polynomials [11] of $u_0, u_1, u_2, ..., u_n$ and it can be calculated by formula given below

\[ A_n = \frac{1}{n!} \frac{d^n}{d\lambda^n} \left[ N \left( \sum_{i=0}^{\infty} \lambda^i u_i \right) \right]_{\lambda=0}, \quad n = 0, 1, 2, .... \]  

(2.9)

Using Eq. (2.7) and Eq. (2.8) in Eq. (2.6) we will get

\[ \mathcal{L} \left[ \sum_{n=0}^{\infty} u_n(x, t) \right] = \frac{f(x)}{s} + \frac{g(x)}{s^2} + \frac{1}{s^2} \mathcal{L} \left[ h(x, t) \right] - \frac{1}{s^2} \mathcal{L} \left[ Ru(x, t) \right] - \frac{1}{s^2} \mathcal{L} \left[ \sum_{n=0}^{\infty} A_n \right], \]
\[
\sum_{n=0}^{\infty} \mathcal{L} \left[ u_n(x, t) \right] = \frac{f(x)}{s} + \frac{g(x)}{s^2} + \frac{1}{s^2} \mathcal{L} \left[ h(x, t) \right] - \frac{1}{s^2} \mathcal{L} \left[ Ru(x, t) \right] - \frac{1}{s^2} \mathcal{L} \left[ \sum_{n=0}^{\infty} A_n \right]
\]  

(2.10)

On comparing both sides of the Eq. (2.10) we have

\[
\mathcal{L} \left[ u_0(x, t) \right] = \frac{f(x)}{s} + \frac{g(x)}{s^2} + \frac{1}{s^2} \mathcal{L} \left[ h(x, t) \right] = K(x, s),
\]

(2.11)

\[
\mathcal{L} \left[ u_1(x, t) \right] = -\frac{1}{s^2} \mathcal{L} \left[ Ru_0(x, t) \right] - \frac{1}{s^2} \mathcal{L} \left[ A_0 \right],
\]

(2.12)

\[
\mathcal{L} \left[ u_2(x, t) \right] = -\frac{1}{s^2} \mathcal{L} \left[ Ru_1(x, t) \right] - \frac{1}{s^2} \mathcal{L} \left[ A_1 \right].
\]

(2.13)

In general, the recursive relation is given by

\[
\mathcal{L} \left[ u_{n+1}(x, t) \right] = -\frac{1}{s^2} \mathcal{L} \left[ Ru_n(x, t) \right] - \frac{1}{s^2} \mathcal{L} \left[ A_n \right], \quad n \geq 1.
\]

(2.14)

Applying inverse Laplace transform to Eq. (2.11) − (2.14), so our required recursive relation is given below

\[
u_0(x, t) = K(x, t),
\]

(2.15)

\[
u_{n+1}(x, t) = -\mathcal{L}^{-1} \left[ \frac{1}{s^2} \mathcal{L} \left[ Ru_n(x, t) \right] + \frac{1}{s^2} \mathcal{L} \left[ A_n \right] \right], \quad n \geq 0,
\]

(2.16)

where \( K(x, t) \) represent the term arising from source term and prescribe initial conditions. Now first of all we applying Laplace transform of the terms on the right hand side of Eq. (2.16) then applying inverse Laplace transform we get the values of \( u_1, u_2, \ldots, u_n \) respectively.

To apply this modification, we assume that \( K(x, t) \) can be divided into the sum of two parts namely \( K_0(x, t) \) and \( K_1(x, t) \), therefore we get

\[
K(x, t) = K_0(x, t) + K_1(x, t).
\]

(2.17)

Under this assumption, we propose a slight variation only in the components \( u_0, u_1 \). The variation we propose is that only the part \( K_0(x, t) \) be assigned to the \( u_0 \), whereas the remaining part \( K_1(x, t) \) be combined with the other terms.
given in Eq. (2.16) to define $u_1$. In view of these suggestion, we formulate the modified recursive algorithm as follows:

$$u_0(x, t) = K_0(x, t),$$
$$u_1(x, t) = K_1(x, t) - \mathcal{L}^{-1} \left[ \frac{1}{s^2} \mathcal{L} [Ru_0(x, t)] + \frac{1}{s^2} \mathcal{L} [A_0] \right],$$
$$u_{n+1}(x, t) = -\mathcal{L}^{-1} \left[ \frac{1}{s^2} \mathcal{L} [Ru_n(x, t)] + \frac{1}{s^2} \mathcal{L} [A_n] \right], \quad n \geq 1.$$  

The solution through the modified Adomian decomposition method is highly depend upon the choice of $K_0(x, t)$ and $K_1(x, t)$.

### 3 Applications

To illustrate this method for nonlinear partial differential equations we take three examples in this section. First of all we solve these nonlinear partial differential equations with LDM and then solve these examples with the help of MLDM.

#### 3.1 Example 1

Consider a nonlinear partial differential equation [18]

$$\frac{\partial u(x, t)}{\partial t} + uu_x = x + xt^2,$$

with initial conditions

$$u(x, 0) = 0.$$  

Applying the Laplace transform (denoted by $\mathcal{L}$) we have

$$su(x, s) - u(x, 0) = \mathcal{L} \left[ x + xt^2 \right] - \mathcal{L} [uu_x],$$

Using initial conditions Eqs.(3.2) becomes

$$u(x, s) = \frac{x}{s^2} + \frac{x^2}{s^4} + \frac{1}{s} \mathcal{L} [uu_x].$$

Applying inverse Laplace transform we get

$$u(x, t) = xt + \frac{xt^3}{3} - \mathcal{L}^{-1} \left[ \frac{1}{s} \mathcal{L} [uu_x] \right].$$
We decompose the solution as an infinite sum given below

\[ u = \sum_{n=0}^{\infty} u_n(x, t). \] (3.6)

The nonlinear term is handled with the help of Adomian polynomials [11] given below

\[ uu_x = \sum_{n=0}^{\infty} A_n(u). \] (3.7)

Using Eq. (3.6) − (3.7) in equation (3.5) we get

\[ \sum_{n=0}^{\infty} u_n(x, t) = xt + \frac{xt^3}{3} - \mathcal{L}^{-1} \left[ \frac{1}{s} \mathcal{L} \left[ \sum_{n=0}^{\infty} A_n(u) \right] \right]. \] (3.8)

The recursive relation is given below

\[ u_0(x, t) = xt + \frac{xt^3}{3}, \] (3.9)

\[ u_1(x, t) = -\mathcal{L}^{-1} \left[ \frac{1}{s} \mathcal{L} \left[ A_0(u) \right] \right], \] (3.10)

\[ u_{n+1}(x, t) = -\mathcal{L}^{-1} \left[ \frac{1}{s} \mathcal{L} \left[ \sum_{n=0}^{\infty} A_n(u) \right] \right], n \geq 1. \] (3.11)

The other components of the solution can easily found by using above recursive relation

\[ u_1(x, t) = -\mathcal{L}^{-1} \left[ \frac{1}{s} \mathcal{L} \left[ \frac{uu_0}{3} \right] \right], \]

\[ = -\frac{xt^3}{3} - \frac{xt^7}{63} - \frac{2xt^5}{15}, \]

\[ u_2(x, t) = -\mathcal{L}^{-1} \left[ \frac{1}{s} \mathcal{L} \left[ A_1(u) \right] \right], \]

\[ = -\mathcal{L}^{-1} \left[ \frac{1}{s} \mathcal{L} \left[ \frac{uu_0}{3} + uu_1 \right] \right], \]

\[ = -\mathcal{L}^{-1} \left[ \frac{1}{s} \mathcal{L} \left[ \left( \frac{-xt^3}{3} - \frac{xt^7}{63} \right) \left( t + \frac{t^3}{3} \right) + \left( xt + \frac{xt^3}{3} \right) \left( -\frac{t^3}{3} - \frac{t^7}{63} - \frac{2t^5}{15} \right) \right] \right], \]

\[ = -\mathcal{L}^{-1} \left[ \frac{1}{s} \mathcal{L} \left[ \frac{2xt^4}{3} - \frac{2xt^6}{9} - \frac{2xt^8}{63} - \frac{2xt^{10}}{189} - \frac{4xt^6}{45} - \frac{4xt^8}{45} \right] \right], \]

\[ = \frac{2xt^5}{15} + \frac{22xt^7}{315} + \frac{38xt^9}{2835} + \frac{2xt^{11}}{2079}, \]

\[ u(x, t) = \sum_{n=0}^{\infty} u_n(x, t) = xt. \] (3.14)
We consider same nonlinear partial differential equation \[18\] and solve it by using modified Laplace decomposition method

\[
\frac{\partial u(x, t)}{\partial t} + uu_x = x + xt^2, \tag{3.15}
\]

with initial conditions

\[u(x, 0) = 0.\] \tag{3.16}

Applying the Laplace transform we have

\[su(x, s) - u(x, 0) = \mathcal{L}[x + xt^2] - \mathcal{L}[uu_x],\] \tag{3.17}

Using initial conditions Eqs.(3.17) becomes

\[u(x, s) = \frac{x}{s^2} + \frac{x^2}{s^4} - \frac{1}{s} \mathcal{L}[uu_x].\] \tag{3.18}

Applying inverse Laplace transform we get

\[u(x, t) = xt + \frac{xt^3}{3} - \mathcal{L}^{-1}\left[\frac{1}{s} \mathcal{L}[uu_x]\right].\] \tag{3.19}

We decompose the solution as an infinite sum given below

\[u = \sum_{n=0}^{\infty} u_n(x, t).\] \tag{3.20}

The nonlinear term is handled with the help of Adomian polynomials \[11\] given below

\[uu_x = \sum_{n=0}^{\infty} A_n(u).\] \tag{3.21}

Using Eq. (3.20) – (3.21) in equation (3.19) we get

\[\sum_{n=0}^{\infty} u_n(x, t) = xt + \frac{xt^3}{3} - \mathcal{L}^{-1}\left[\frac{1}{s} \mathcal{L}\left[\sum_{n=0}^{\infty} A_n(u)\right]\right].\] \tag{3.22}

The modified recursive relation is given below

\[u_0(x, t) = xt,\] \tag{3.23}

\[u_1(x, t) = \frac{xt^3}{3} - \mathcal{L}^{-1}\left[\frac{1}{s} \mathcal{L}[A_0(u)]\right],\] \tag{3.24}

\[u_{n+1}(x, t) = -\mathcal{L}^{-1}\left[\frac{1}{s} \mathcal{L}\left[\sum_{n=0}^{\infty} A_n(u)\right]\right], \quad n \geq 1.\] \tag{3.25}
where $A_n(u)$ are Adomian polynomials representing the nonlinear terms [11] in above Eq. (3.25). In view of the recursive relations (3.23) – (3.25) we obtained other components as follows

$$u_1(x, t) = \frac{xt^3}{3} - \mathcal{L}^{-1} \left[ \frac{1}{s} \mathcal{L} \left[ \sum_{n=0}^{\infty} A_0(u) \right] \right],$$

$$= \frac{xt^3}{3} - \mathcal{L}^{-1} \left[ \frac{1}{s} \mathcal{L} [u_0u_0] \right],$$

$$= \frac{xt^3}{3} - \mathcal{L}^{-1} \left[ \frac{1}{s} \mathcal{L} [(xt)(t)] \right],$$

$$= \frac{xt^3}{3} - x \mathcal{L}^{-1} \left[ \frac{2!}{s^4} \right],$$

$$= \frac{xt^3}{3} - \frac{2!x}{3!} \mathcal{L}^{-1} \left[ \frac{3!}{s^4} \right],$$

$$= \frac{xt^3}{3} - \frac{xt^3}{3},$$

$$u_1(x, t) = 0, \quad (3.26)$$

$$u_{n+1}(x, t) = 0, \quad n \geq 1. \quad (3.27)$$

In view of above modified recursive relation we get exact solution

$$u(x, t) = \sum_{n=0}^{\infty} u_n(x, t) = xt. \quad (3.28)$$

### 3.2 Example 2

Consider another nonlinear partial differential equation [18]

$$\frac{\partial^2 u(x, y)}{\partial x^2} - u_xu_{yy} = -x + u, \quad (3.29)$$

with initial conditions

$$u(x, 0) = \sin y, \quad (3.30)$$

$$u_x(x, 0) = 1. \quad (3.31)$$

Applying the Laplace transform we get

$$s^2 u(s, y) - su(0, y) - u_x(0, y) = \mathcal{L} [-x + u] + \mathcal{L}[u_xu_{yy}],$$

$$s^2 u(s, y) - s \sin y - 1 = -\frac{1}{s} + \mathcal{L}[u + u_xu_{yy}],$$

$$u(s, y) = \frac{1}{s^2} + \frac{\sin y}{s} - \frac{1}{s^4} + \frac{1}{s^3} \mathcal{L}[u + u_xu_{yy}]. \quad (3.32)$$
Applying inverse transform we get
\[ u(x, y) = x + \sin y - \frac{x^3}{3!} + \mathcal{L}^{-1}\left[ \frac{1}{s^2} \mathcal{L}[u + u_x u_{yy}] \right], \quad (3.33) \]

Applying the same procedure as in previous example we arrive at modified recursive relation given below
\[ u_0(x, y) = x + \sin y, \quad (3.34) \]
\[ u_1(x, y) = -\frac{x^3}{3!} + \mathcal{L}^{-1}\left[ \frac{1}{s^2} \mathcal{L}\left[ \sum_{n=0}^{\infty} u_n(x, t) + \sum_{n=0}^{\infty} B_0(u) \right] \right], \quad (3.35) \]
\[ u_{n+1}(x, y) = \mathcal{L}^{-1}\left[ \frac{1}{s^2} \mathcal{L}[u_n + \sum_{n=0}^{\infty} B_n(u)] \right], \quad n \geq 1. \quad (3.36) \]

where \( B_n(u) \) is a Adomian polynomials [11] representing the nonlinear terms in above Eq. (3.36). The other components of the series solution can be calculated by using above recursive relation
\[ u_0(x, y) = x + \sin y, \quad (3.37) \]
\[ u_1(x, y) = -\frac{x^3}{3!} + \mathcal{L}^{-1}\left[ \frac{1}{s^2} \mathcal{L}[u_0 + \sum_{n=0}^{\infty} B_0(u)] \right], \]
\[ = -\frac{x^3}{3!} + \mathcal{L}^{-1}\left[ \frac{1}{s^2} \mathcal{L}[u_0 + u_0 u_{0yy}] \right], \]
\[ = -\frac{x^3}{3!} + \mathcal{L}^{-1}\left[ \frac{1}{s^2} \mathcal{L}[x + \sin y + u_0 u_{0yy}] \right], \]
\[ = -\frac{x^3}{3!} + \mathcal{L}^{-1}\left[ \frac{1}{s^2} \mathcal{L}[x + \sin y - \sin y] \right], \]
\[ = -\frac{x^3}{3!} + \mathcal{L}^{-1}\left[ \frac{1}{s^2} \mathcal{L}[x] \right], \]
\[ = -\frac{x^3}{3!} + \frac{x^3}{3!}, \]
\[ = 0, \quad (3.38) \]
\[ u_{n+1}(x, t) = 0, \quad n \geq 1. \quad (3.39) \]

The total solution of the above problem is given below
\[ u(x, t) = \sum_{n=0}^{\infty} u_n(x, t) = x + \sin y. \quad (3.40) \]
3.3 Example 3

We consider system of coupled nonlinear partial differentials [18]

\[
\begin{align*}
\frac{\partial u(x, t)}{\partial t} + vu_x + u &= 1, \quad t > 0 \quad (3.41) \\
\frac{\partial v(x, t)}{\partial t} - uv_x - v &= 1, \quad (3.42)
\end{align*}
\]

with initial conditions

\[
\begin{align*}
u(x, 0) &= e^x, \quad (3.43) \\
v(x, 0) &= e^{-x}. \quad (3.44)
\end{align*}
\]

To solve Eqs. (3.41) – (3.42), first of all we will apply Laplace transform (denoted by \(\mathcal{L}\)) and using given initial conditions we get

\[
\begin{align*}
sv(x, s) - v(x, 0) &= \mathcal{L} [1 + uv_x + v], \quad (3.46)
\end{align*}
\]

Using given initial conditions Eqs. (3.45) – (3.46) becomes

\[
\begin{align*}
u(x, s) &= \frac{e^x}{s} + \frac{1}{s^2} - \frac{1}{s} \mathcal{L} [vu_x + u], \quad (3.47) \\
v(x, s) &= \frac{e^{-x}}{s} + \frac{1}{s^2} + \mathcal{L} [uv_x + v] \quad (3.48)
\end{align*}
\]

We decompose the solution as infinite sum given below

\[
\begin{align*}
u &= \sum_{n=0}^{\infty} u_n(x, t), \quad (3.49) \\
v &= \sum_{n=0}^{\infty} v_n(x, t). \quad (3.50)
\end{align*}
\]

Using Eqs.(3.49) – (3.50) we have

\[
\sum_{n=0}^{\infty} u_n(x, t) = e^x + t - \mathcal{L}^{-1} \left[ \frac{1}{s} \mathcal{L} \left[ \sum_{n=0}^{\infty} C_n(u, v) + \sum_{n=0}^{\infty} u_n(x, t) \right] \right], \quad (3.51)
\]
\[ \sum_{n=0}^{\infty} v_n(x, t) = e^{-x} + t + \mathcal{L}^{-1} \left[ \frac{1}{s} \mathcal{L} \left( \sum_{n=0}^{\infty} D_n(u, v) + \sum_{n=0}^{\infty} v_n(x, t) \right) \right], \quad (3.52) \]

where \( C_n(u, v), D_n(u, v) \) are Adomian polynomials [11]. Now our required modified recursive relations are given by

\[
\begin{align*}
  u_0(x, t) & = e^{x}, \\
  u_1(x, t) & = t - \mathcal{L}^{-1} \left[ \frac{1}{s} \mathcal{L} \left( C_0(u, v) + u_0 \right) \right], \\
  u_{n+1}(x, t) & = -\mathcal{L}^{-1} \left[ \frac{1}{s} \mathcal{L} \left( C_n(u, v) + u_n \right) \right], \quad n \geq 1. \quad (3.53)
\end{align*}
\]

\[
\begin{align*}
  v_0(x, t) & = e^{-x}, \\
  v_1(x, t) & = t + \mathcal{L}^{-1} \left[ \frac{1}{s} \mathcal{L} \left( D_0(u, v) + v_0 \right) \right], \\
  v_{n+1}(x, t) & = \mathcal{L}^{-1} \left[ \frac{1}{s} \mathcal{L} \left( D_n(u, v) + v_n \right) \right], \quad n \geq 1. \quad (3.54)
\end{align*}
\]

Where \( C_n(u, v), D_n(u, v) \) are Adomian polynomials representing nonlinear terms [11]. Applying the same procedure as describe in previous example,
we get

\[ u_1(x,t) = t - \mathcal{L}^{-1} \left[ \frac{1}{s} \mathcal{L} \left[ C_0(u,v) + u_0 \right] \right], \]
\[ = t - \mathcal{L}^{-1} \left[ \frac{1}{s} \mathcal{L} \left[ v_0 u_{0x} + u_0 \right] \right], \]
\[ = t - \mathcal{L}^{-1} \left[ \frac{1}{s} \mathcal{L} \left[ e^x e^{-x} + e^x \right] \right], \]
\[ = t - \mathcal{L}^{-1} \left[ \frac{1}{s} \mathcal{L} \left[ 1 + e^x \right] \right], \]
\[ = t - \mathcal{L}^{-1} \left[ \frac{1}{s^2} + \frac{e^x}{s^2} \right], \]
\[ = t - t - te^x, \]
\[ = -te^x, \] (3.55)

\[ v_1(x,t) = t + \mathcal{L}^{-1} \left[ \frac{1}{s} \mathcal{L} \left[ D_0(u,v) + v_0 \right] \right], \]
\[ = t + \mathcal{L}^{-1} \left[ \frac{1}{s} \mathcal{L} \left[ u_0 v_{0x} + v_0 \right] \right], \]
\[ = t + \mathcal{L}^{-1} \left[ \frac{1}{s} \mathcal{L} \left[ -e^x e^{-x} + e^{-x} \right] \right], \]
\[ = t + \mathcal{L}^{-1} \left[ \frac{1}{s} \mathcal{L} \left[ -1 + e^{-x} \right] \right], \]
\[ = t + \mathcal{L}^{-1} \left[ \frac{1}{s} \mathcal{L} \left[ -1 + e^{-x} \right] \right], \]
\[ = t + \mathcal{L}^{-1} \left[ \frac{1}{s^2} + \frac{e^{-x}}{s^2} \right], \]
\[ = t - t + te^{-x}, \]
\[ = te^{-x}, \] (3.56)

\[ u_2(x,t) = -\mathcal{L}^{-1} \left[ \frac{1}{s} \mathcal{L} \left[ C_1(u,v) + u_1 \right] \right], \]
\[ = -\mathcal{L}^{-1} \left[ \frac{1}{s} \mathcal{L} \left[ v_1 u_{0x} + v_0 u_{1x} + u_1 \right] \right], \]
\[ = -\mathcal{L}^{-1} \left[ \frac{1}{s} \mathcal{L} \left[ t e^{-x} e^x + e^{-x} (-te^x) + te^x \right] \right], \]
\[ = \frac{i^2 e^x}{2!}, \] (3.57)

\[ v_2(x,t) = \mathcal{L}^{-1} \left[ \frac{1}{s} \mathcal{L} \left[ D_1(u,v) + v_1 \right] \right], \]
\[ = \mathcal{L}^{-1} \left[ \frac{1}{s} \mathcal{L} \left[ u_1 v_{0x} + u_0 v_{1x} + v_1 \right] \right], \]
\[ = \mathcal{L}^{-1} \left[ \frac{1}{s} \mathcal{L} \left[ u_1 v_{0x} + u_0 v_{1x} + v_1 \right] \right], \]
\[ = \frac{i^2 e^{-x}}{2!}. \] (3.58)
In view of Eqs. (3.53) – (3.58), the series solution is
\[
\begin{align*}
    u(x, t) &= e^x - te^x + \frac{t^2 e^x}{2!} - \ldots, \\
             &= e^x \left[ 1 - t + \frac{t^2}{2!} - \ldots \right], \\
             &= e^{x-t}.
    \end{align*}
\]

(3.59)

\[
\begin{align*}
    v(x, t) &= e^{-x} + te^{-x} + \frac{t^2 e^{-x}}{2!} + \ldots, \\
             &= e^{-x} \left[ 1 + t + \frac{t^2}{2!} + \ldots \right], \\
             &= e^{t-x}.
    \end{align*}
\]

(3.60)

With the help of this modification we are able to get exact solution very easily.

4 Conclusion

In this work, we carefully proposed a reliable modification of Laplace decomposition method. We solved three nonlinear partial differential equations with initial conditions. Example 1 has been solved by two ways, first with Laplace decomposition and secondly with modified Laplace decomposition method. It is evident from this example that ”noise terms” appear in the components of the solution series obtained by Laplace decomposition method. While solution obtained by modified Laplace decomposition does not contain any noise terms. In similar manners examples 2 and 3, which is nonlinear coupled partial differential equation, are solved. This modified technique has been shown to computationally efficient in these examples that are important to researchers in the field of applied sciences. In addition, the modified Laplace decomposition method may give the exact solutions for nonlinear partial or coupled partial differential equations.

References


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