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Uniform Approximation of a Class of Multivariable Trigonometric Interpolation Polynomials in Euclidean Space

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Abstract

To improve the uniform convergence of the classical Lagrange operators of several variables, we construct a new operator with a class of summation factors. It is proved that the new operator converges to arbitrary continuous functions with period 2π in Euclidean space and that it has the best convergence order.

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1. Formulation and Result

Here we will study the approximation problems of some triangle interpolation operators with a class of summation factors in Euclidean space.

Let $E(\Omega)$ be the known $n-$dimensional Euclidean space, where $\Omega = \{(x_1, x_2, \cdots, x_n) | 0 \leq x_i \leq 2\pi, i = 1, 2, \cdots, n\}$ and let $z = f(x_1, x_2, \cdots, x_n)$, $(x_1, x_2, \cdots, x_n) \in \Omega$ be any real-valued continuous functions with period 2π in

$E(\Omega)$. Moreover, the 2−norm (or Euclidean norm) and the continuity modulus are respectively defined by

$$\| X - Y \|_2 = \left(\sum_{i=1}^{n} (x_i - y_i)^2 \right)^{1/2}$$

and

$$\omega(f, \delta) = \max_{\|X-Y\|_2 \le \delta} |f(X) - f(Y)|,$$

where $X, Y \in \Omega$. It is easy to show that

$$| f(X) - f(Y) | \le (1 + \frac{\| X - Y \|_2}{\delta}) \omega(f, \delta)$$

To describe the approximation problem conveniently, we only consider the corresponding problem in the 2−dimensional Euclidean space. Moreover, the results obtained in this work can be generalized to the n−dimensional Euclidean space.

Let $(x_1^{(j)}, x_2^{(k)}) = \left(\frac{(2j+1)\pi}{2s}, \frac{(2k+1)\pi}{2t} \right)$ be the interpolation nodes, where $j = 0, 1, 2, \cdots, 2s-1$ and $k = 0, 1, 2, \cdots, 2t-1$, then the Lagrange trigonometric interpolation operator of two variables can be written as

$$L_{st}(f; x_1, x_2) = \frac{1}{ST} \sum_{j=0}^{2s-1} \sum_{k=0}^{2t-1} f(x_1^{(j)}, x_2^{(k)}) \left(1 + 2 \sum_{\alpha=1}^{s} \cos \alpha (x - x_1^{(j)}) \right.$$

$$\left. + 2 \sum_{\beta=1}^{t} \cos \beta (x - x_2^{(k)}) + 4 \sum_{\alpha=1}^{s} \sum_{\beta=1}^{t} \cos \alpha (x - x_1^{(j)}) \cos \beta (x - x_2^{(k)}) \right). \quad (1)$$

where $S = 1 + 2s, T = 1 + 2t$. Obviously, Eq.(1) satisfies $L_{st}(f; x_1^{(j)}, x_2^{(k)}) = f(x_1^{(j)}, x_2^{(k)})$. However, it is known that $L_{st}(f; x_1, x_2)$ does not converge uniformly to arbitrary continuous functions in the 2−dimensional Euclidean space[1]. In this work, To improve the uniform convergence, we introduce a class of summation factors and construct a new interpolation operator with a class of summation factor. The summation factor is denoted by

$$\gamma_{\alpha,\beta} = \frac{1}{4}(1 - 4\cos\alpha - 4\cos\beta + \cos 2\alpha + \cos 2\beta - 4\cos 2\alpha \cos\beta$$

$$-4\cos\alpha\cos 2\beta + 16\cos\alpha\cos\beta + \cos 2\alpha \cos 2\beta), \quad (2)$$

where $\alpha = 1, 2, \cdots, s$ and $\beta = 1, 2, \cdots, t$.

The operator constructed in this paper is given by

$$H_{st}(f;x_1,x_2)=\frac{1}{ST}\sum_{j=0}^{2s-1}\sum_{k=0}^{2t-1}f(x_1^{(j)},x_2^{(k)})\Bigg(1+2\sum_{\alpha=1}^{s}\gamma_{\alpha,0}\cos\alpha(x-x_1^{(j)})$$

$$+2\sum_{\beta=1}^{t}\gamma_{0,\beta}\cos\beta(x-x_2^{(k)})$$

$$+4\sum_{\alpha=1}^{s}\sum_{\beta=1}^{t}\gamma_{\alpha,\beta}\cos\alpha(x-x_1^{(j)})\cos\beta(x-x_2^{(k)})\Bigg). \tag{3}$$

For the operator $H_{st}(f;x_1,x_2)$, we have the following result.
Theorem If $f(x_1,x_2)\in C_{\Omega}^{u,v}, 0\le u,v\le 3$, then

$$|H_{st}(f;x_1,x_2)-f(x_1,x_2)|=O\Bigg(E_{st}^{*}(f)+(\frac{\pi}{2s})^u\omega(f_{x_1^u}^{(u)},\frac{\pi}{2s})+$$

$$(\frac{\pi}{2t})^v\omega(f_{x_2^v}^{(v)},\frac{\pi}{2t})+(\frac{\pi}{2s})^u(\frac{\pi}{2t})^v\omega(f_{x_1^u x_2^v}^{(u+v)},((\frac{\pi}{2s})^2+(\frac{\pi}{2t})^2)^{1/2}\Bigg), \tag{4}$$

where "O" is independent of $s,t,x_1,x_2,f,\cdots,f_{x_1^u x_2^v}^{(u+v)}$, $E_{st}^{*}(f)$ is the minimum deviation with $f(x_1,x_2)$, $\omega(f_{x_1^u x_2^v}^{(u+v)},\delta)$ is the $(u+v)$-*th* modulus of continuity of $f(x_1,x_2)$.

2. Proof of theorem

To prove the theorem, we need some preparation work.

If the trigonometric polynomial $p(x_1,x_2)$ has the minimum deviation with $f(x_1,x_2)$, then the following expressions are valid, i.e.,

$$|p(x_1,x_2)-f(x_1,x_2)|\le E_{st}^{*}(f)^{[2]}, \tag{5}$$

$$H_{st}(p;x_1,x_2)=p(x_1,x_2)-\frac{1}{4}\sum_{i=0}^{4}(-1)^i\begin{pmatrix}4\\i\end{pmatrix}p(x_1-\frac{(2-i)\pi}{2s}),x_2)$$

$$-\frac{1}{4}\sum_{l=0}^{4}(-1)^l\begin{pmatrix}4\\l\end{pmatrix}p(x_1,x_2-\frac{(2-l)\pi}{2t}))$$

$$+\frac{1}{16}\sum_{i=0}^{4}\sum_{l=0}^{4}(-1)^{i+l}\begin{pmatrix}4\\i\end{pmatrix}\begin{pmatrix}4\\l\end{pmatrix}p(x_1-\frac{(2-i)\pi}{2s},x_2-\frac{(2-l)\pi}{2t})). \tag{6}$$

Next we divide the operator $H_{st}(f;x_1,x_2)$ as follows.

$$H_{st}(f;x_1,x_2)-f(x_1,x_2)=\sum_{a=1}^{8}e_a, \tag{7}$$

where

$$e_1=H_{st}(f-p;x_1,x_2), \tag{8}$$

$$e_2=p(x_1,x_2)-f(x_1,x_2), \tag{9}$$

$$e_3=-\frac{1}{4}\sum_{i=0}^{4}(-1)^i\left(\begin{array}{c}4\\ i\end{array}\right)(p(x_1-\frac{(2-i)\pi}{2s},x_2)-f(x_1-\frac{(2-i)\pi}{2s},x_2)), \tag{10}$$

$$e_4=-\frac{1}{4}\sum_{l=0}^{4}(-1)^l\left(\begin{array}{c}4\\ l\end{array}\right)(p(x_1,x_2-\frac{(2-l)\pi}{2t})-f(x_1,x_2-\frac{(2-l)\pi}{2t})), \tag{11}$$

$$\begin{aligned}e_5=\frac{1}{16}\sum_{i=0}^{4}\sum_{l=0}^{4}(-1)^{i+l}\left(\begin{array}{c}4\\ i\end{array}\right)\left(\begin{array}{c}4\\ l\end{array}\right)(p(x_1-\frac{(2-i)\pi}{2s},x_2-\frac{(2-l)\pi}{2t})\\ -f(x_1-\frac{(2-i)\pi}{2s},x_2-\frac{(2-l)\pi}{2t})),\end{aligned} \tag{12}$$

$$e_6=-\frac{1}{4}\sum_{i=0}^{4}(-1)^i\left(\begin{array}{c}4\\ i\end{array}\right)f(x_1-\frac{(2-i)\pi}{2s},x_2), \tag{13}$$

$$e_7=-\frac{1}{4}\sum_{l=0}^{4}(-1)^l\left(\begin{array}{c}4\\ l\end{array}\right)f(x_1,x_2-\frac{(2-l)\pi}{2t}), \tag{14}$$

$$e_8=\frac{1}{16}\sum_{i=0}^{4}\sum_{l=0}^{4}(-1)^{i+l}\left(\begin{array}{c}4\\ i\end{array}\right)\left(\begin{array}{c}4\\ l\end{array}\right)(f(x_1-\frac{(2-i)\pi}{2s},x_2-\frac{(2-l)\pi}{2t}). \tag{15}$$

Using Eq.(5) and the following equation[2]

$$\begin{aligned}\Bigg|\frac{1}{ST}\sum_{j=0}^{2s-1}\sum_{k=0}^{2t-1}\Bigg(1+2\sum_{\alpha=1}^{s}\gamma_{\alpha,0}\cos\alpha(x-x_1^{(j)})+2\sum_{\beta=1}^{t}\gamma_{0,\beta}\cos\beta(x-x_2^{(k)})\\ +4\sum_{\alpha=1}^{s}\sum_{\beta=1}^{t}\gamma_{\alpha,\beta}\cos\alpha(x-x_1^{(j)})\cos\beta(x-x_2^{(k)})\Bigg)\Bigg|=O(1),\end{aligned} \tag{16}$$

we know that

$$e_a\le E^*_{st}(f),a=1\sim 5 \tag{17}$$

Using the relation between derivative and difference, it is not difficult to show that[3]

$$e_6=O\left((\frac{\pi}{2s})^u\omega(f^{(u)}_{x_1^u},\frac{\pi}{2s})\right),e_7=O\left((\frac{\pi}{2t})^v\omega(f^{(v)}_{x_2^v},\frac{\pi}{2t})\right),$$

$$e_8 = O\left((\frac{\pi}{2s})^u (\frac{\pi}{2t})^v \omega(f^{(u+v)}_{x_1^u x_2^v}, ((\frac{\pi}{2s})^2 + \frac{\pi}{2t})^2)^{1/2} \right) \tag{18}$$

Combining the estimation expressions $e_1 \sim e_8$, we know that the theorem is valid.

Remark. Here we also know that the following expression is valid uniformly in $C(\Omega)$, namely,

$$\lim_{s,t\to\infty} H_{st}(f; x_1, x_2) = f(x_1, x_2)$$

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