Applied Mathematical Sciences, Vol. 4, 2010, no. 31, 1535 - 1545

A Family of Fractional Integrals Pertaining to Multivariable I-Function

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Abstract

The object of this paper is to derive an Eulerian integral and a main theorem based upon the fractional operator associated with $\bar{H}$ -function [3] and multivariable I-function [4] which provide unification and extension of numerous results in the theory of fractional calculus of special functions in one and more variables. Some interesting special cases are also given.

Mathematics Subject Classification: 26A33, 33C20, 44A10

Keywords: Fractional integral, Eulerian integral, $\bar{H}$ -function, multivariable I-function, Riemann-Liouville operator, Lauricella function

1 Introduction

In recent years several authors namely Srivastava and Hussain [11], Saxena and Saigo [8], Saigo and Saxena [6] have been established certain fractional integral formulas deduced from Eulerian integrals.
The Riemann-Liouville operator of fractional integration $R^m f$ of order m is defined by

$$ {}_xD_y^{-m}[f(y)] = \frac{1}{\Gamma(m)} \int_x^y (y-t)^{m-1} f(t)dt, \tag{1} $$

for $Re(m) > 0$ and a constant x. An equivalent form of Beta function is [2, p.10, eq.(13)]:

$$\int_{m}^{n} (t-m)^{a-1}(n-t)^{b-1}\,dt = (n-m)^{a+b-1}B(a,b), \tag{2}$$

where $m, n \in R(x < y)$, $Re(a) > 0$, $Re(b) > 0$.
Making use of [2, p.62, eq. (15)], we have

$$(pt+q)^{\alpha} = (xp+q)^{\alpha}\left[1+\tfrac{p(t-x)}{xp+q}\right]^{\alpha}$$

$$= \frac{(xp+q)^{\alpha}}{\Gamma(-\alpha)}\frac{1}{(2\pi i)}\int_{-i\infty}^{+i\infty}\Gamma(-\beta)\,\Gamma(\beta-\alpha)\left[\frac{p(t-x)}{xp+q}\right]^{\beta}d\beta, \tag{3}$$

where $i = \sqrt{-1}$; $p, q, \alpha \in C$; $x, t \in R$; $\left|\arg\left(\frac{p}{xp+q}\right)\right| < \pi$ and the path of integration is indented if necessary in such a manner so as to separate the poles of $\Gamma(-\beta)$ from those of $\Gamma(\beta-\alpha)$.

The multivariable I-function is defined and represented in the following manner [4]:

$$I[z_1,...,z_r] = I^{0,n_2;\,0,n_3\,;...;\,0,n_r\,:\,(m',n')\,;...;\,(m^{(r)},n^{(r)})}_{p_2,q_2;\,p_3,q_3\,;...;\,p_r,q_r\,:\,(p',q')\,;...;\,(p^{(r)},q^{(r)})}$$
$$\left[\begin{array}{c|c} z_1 & (a_{2j};\alpha'_{2j},\alpha''_{2j});(a_{3j};\alpha'_{3j},\alpha''_{3j});...;(a_{rj};\alpha'_{rj},...,\alpha^{(r)}_{rj});(a'_j,\alpha'_j)_{1,p'};...;(a^{(r)}_j,\alpha^{(r)}_j)_{1,p^{(r)}} \\ \vdots & \\ z_r & (b_{2j};\beta'_{2j},\beta''_{2j});(b_{3j};\beta'_{3j},\beta''_{3j});...;(b_{rj};\beta'_{rj},...,\beta^{(r)}_{rj});(b'_j,\beta'_j)_{1,q'};...;(b^{(r)}_j,\beta^{(r)}_j)_{1,q^{(r)}} \end{array}\right] \tag{4}$$
$$= \frac{1}{(2\pi i)^r}\int_{L_1}...\int_{L_r}\varphi_1(\xi_1)...\varphi_r(\xi_r)\,\psi(\xi_1,...,\xi_r)\,z_1^{\xi_1}...z_r^{\xi_r}\,d\xi_1...d\xi_r,$$

where $i = \sqrt{(-1)}$.
For convergence conditions and other details of multivariable I-function, see Prasad [4]. The Lauricella function $F_D^{(h)}$ is defined in the following integral form

$$\frac{\Gamma(a)\,\Gamma(b_1)...\Gamma(b_h)}{\Gamma(c)}F_D^h[a,b_1,...,b_h;\,c;x_1,...,x_h]$$
$$= \frac{1}{(2\pi i)^h}\int_{-i\infty}^{+i\infty}...\int_{-i\infty}^{+i\infty}\frac{\Gamma(a+\xi_1+...+\xi_h)\,\Gamma(b_1+\xi_1)...\Gamma(b_h+\xi_h)}{\Gamma(c+\xi_1+...+\xi_h)} \tag{5}$$
$$.\Gamma(-\xi_1)...\,\Gamma(-\xi_h)\,(-x_1)^{\xi_1}...(-x_h)^{\xi_h}\,d\xi_1...d\xi_h,$$

where $\max[\,|\arg(-x_1)|,...,|\arg(-x_h)|\,] < \pi$; $c = 0, -1, -2, ...$
To prove the Eulerian integrals, we use the following formula:

$$\int_{x}^{y}(t-x)^{a-1}(y-t)^{b-1}(p_1t+q_1)^{\rho_1}...(p_ht+q_h)^{\rho_h}\,dt$$

$$= (y-x)^{a+b-1}B(a,b)(p_1x+q_1)^{\rho_1}...(p_hx+q_h)^{\rho_h}$$
$$.F_D^{(h)}\left[a,-\rho_1,...,-\rho_h;\,a+b;\,-\frac{(y-x)p_1}{p_1x+q_1},...,-\frac{(y-x)p_h}{p_h+q_h}\right], \tag{6}$$

where $x, y \in R(x < y); p_j, q_j, \rho_j \in C(j = 1,..., h)$;

$$\min[Re(m), Re(n)] > 0 \text{and} \max\left[\left|\frac{(y-x)p_1}{p_1x+q_1}\right|, ..., \left|\frac{(y-x)p_h}{p_hx+q_h}\right|\right] < 1 .$$

Making use of the results (2), (3) and (5), we can prove the formula given in (6). For h = 1 and h = 2, we get the known results [5, p.301, entry (2.2.6.1)] and [11, p.81, eq. (3.6)] respectively.

In what follows h is a positive integer and and 0,,0 would mean h zero. The series representation of $\bar{H}$-function [3] is as follows

$$\begin{aligned}\bar{H}^{M,N}_{P,Q}[z] &= \bar{H}^{M,N}_{P,Q}\left[z\left|\begin{matrix}(e_j,E_j;\alpha_j)_{1,N},(e_j,E_j)_{N+1,P}\\(f_j,F_j)_{1,M},(f_j,F_j;\beta_j)_{M+1,Q}\end{matrix}\right.\right]\\&= \sum_{g=1}^{M}\sum_{k=0}^{\infty}\frac{(-1)^k\varphi(\eta_{g,k})}{k!F_g}z^{\eta_{g,k}}\end{aligned} \tag{7}$$

where

$$\phi(\eta_{g,k}) = \frac{\prod\limits_{\substack{j=1\\j\neq g}}^{M}\Gamma(f_j - F_j\eta_{g,k})\prod\limits_{j=1}^{N}\{\Gamma 1 - e_j + E_j\eta_{g,k})\}^{\alpha_j}}{\prod\limits_{j=M+1}^{Q}\{\Gamma(1 - f_j + F_j\eta_{g,k}\}^{\beta_j}\prod\limits_{j=N+1}^{P}\Gamma(e_j - E_j\eta_{g,k})}$$

and

$$\eta_{g,k} = \frac{f_g + k}{F_g}.$$

For convergence conditions and other details of the $\bar{H}$-function see Inayat-Hussain [3].

2 Main Integral

The main integral to be established here is

$$\begin{aligned}&\int_m^n (t-m)^{a-1}(n-t)^{b-1}\left\{\prod_{j=1}^{h}(p_jt+q_j)^{\rho_j}\right\}\\&.\bar{H}^{M,N}_{P,Q}\left[x(t-m)^{\lambda}(n-t)^{\mu}\prod_{j=1}^{h}(p_jt+q_j)^{\sigma_j}\right]\\&.I\left[\begin{matrix}z_1(t-m)^{\gamma_1}(n-t)^{\tau_1}\prod\limits_{j=1}^{h}(p_jt+q_j)^{-c_j'}\\ \vdots\\ z_r(t-m)^{\gamma_r}(n-t)^{\tau_r}\prod\limits_{j=1}^{h}(p_jt+q_j)^{-c_j^{(r)}}\end{matrix}\right]dt\end{aligned}$$

$$
\begin{aligned}
&= \mathrm{G}_1 \sum_{g=1}^{M} \sum_{k=0}^{\infty} \frac{(-1)^k \varphi(\eta_{g,k})}{k!F_g} x^{\eta_{g,k}} \mathrm{G}_2 \\
&.I^{0,n_2\,;\,0,n_3\,;\ldots;\,0,n_r+h+2\,:\,(m',n')\,;\ldots;\,(m^{(r)},n^{(r)})\,;\,(1,0)\,;\ldots;\,(1,0)}_{p_2,q_2\,;\,p_3,q_3\,;\ldots;\,p_r+h+2,q_r+h+1\,:\,(p',q')\,;\ldots;\,(p^{(r)},q^{(r)})\,;\,(0,1)\,;\ldots;\,(0,1)} \\
&\left[\begin{matrix} A_1,A_2,A_3,(a_{2j};\alpha'_{2j},\alpha''_{2j})_{1,p_2};(a_{3j};\alpha'_{3j},\alpha''_{3j},\alpha'''_{3j})_{1,p_3};\ldots; \\ (b_{2j};\beta'_{2j},\beta''_{2j})_{1,q_2};(b_{3j};\beta'_{3j},\beta''_{3j},\beta'''_{3j})_{1,q_2};\ldots;(b_{rj},\beta'_{rj},\ldots,\beta^{(r)}_{rj},0,\ldots,0)_{1,q_r}, \end{matrix}\right. \\
&\left.\begin{matrix} (a_{rj};\alpha'_{rj},\ldots,\alpha^{(r)}_{rj},0,\ldots,0);(a'_j,\alpha'_j)_{1,p'};\ldots;(a^{(r)}_j,\alpha^{(r)}_j)_{1,p^{(r)}};-;\ldots;-;R_1 \\ A_4,A_5,(b'_j,\beta'_j)_{1,q'};\ldots;(b^{(r)}_j,\beta^{(r)}_j)_{1,q^{(r)}};(0,1);\ldots;(0,1)\quad;R_2 \end{matrix}\right]
\end{aligned}
\tag{8}
$$

The following are the conditions of the validity of (8):

$$
\begin{aligned}
&(i) m,n \in R(m<n);\gamma_i;\tau_i;c_j^{(i)},\lambda,\mu,\sigma_j \in R^+,\rho_j \in R, \\
&p_j,\ q_j \in C, z_i \in C(i=1,\ldots,r;j=1,\ldots,h); \\
&(ii)\ \max_{1\le j\le h}\left[\left|\frac{(n-m)p_j}{p_jm+q_j}\right|\right] < 1\ ; \\
&(iii)\mathrm{Re}\left[a+\lambda\frac{f_j}{F_j}+\sum_{i=1}^{r}\gamma_i\frac{b_j^{(i)}}{\beta_j^{(i)}}\right] > 0\ \ (j=1,\ldots,m^{(i)}), \\
&\mathrm{Re}\left[b+\mu\frac{f_j}{F_j}+\sum_{i=1}^{r}\tau_i\frac{b_j^{(i)}}{\beta_j^{(i)}}\right] > 0\ \ (j=1,\ldots,m^{(i)}); \\
&(iv)\left|\arg(z_i)\prod_{j=1}^{h}(p_jt+q_j)^{-c_j^{(i)}}\right| < \frac{T_i\pi}{2}\ \ (m\le\ t\le\ n;i=1,\ldots,r),
\end{aligned}
$$

where

$$
\begin{aligned}
T_i =& \sum_{j=1}^{n^{(i)}}\alpha_j^{(i)} - \sum_{j=n^{(i)}+1}^{p^{(i)}}\alpha_j^{(i)} + \sum_{j=1}^{m^{(i)}}\beta_j^{(i)} - \sum_{j=m^{(i)}+1}^{q^{(i)}}\beta_j^{(i)} \\
&+\left(\sum_{j=1}^{n_2}\alpha_{2j}^{(i)} - \sum_{j=n_2+1}^{p_2}\alpha_{2j}^{(i)}\right) + \left(\sum_{j=1}^{n_3}\alpha_{3j}^{(i)} - \sum_{j=n_3+1}^{p_3}\alpha_{3j}^{(i)}\right) + \ldots \\
&+\left(\sum_{j=1}^{n_r}\alpha_{rj}^{(i)} - \sum_{j=n_r+1}^{p_r}\alpha_{rj}^{(i)}\right) + \left(\sum_{j=1}^{q_2}\beta_{2j}^{(i)} + \sum_{j=1}^{q_3}\beta_{3j}^{(i)} + \ldots + \sum_{j=1}^{q_r}\beta_{rj}^{(i)}\right).
\end{aligned}
$$

Here

$$
\begin{aligned}
G_1 &= (n-m)^{a+b-1}\left\{\prod_{j=1}^{h}(p_jm+q_j)^{\rho_j}\right\}, \\
G_2 &= (n-m)^{(\lambda+\mu)\eta_{g,k}}\left\{\prod_{j=1}^{h}(p_jm+q_j)^{\sigma_j\eta_{g,k}}\right\}, \\
A_1 &= (1-a-\lambda\,\eta_{g,k}:\ \gamma_1,\ldots,\gamma_r,\ 1,\ldots,1), \\
A_2 &= (1-\ b-\ \mu\,\eta_{g,k}\ :\ \tau_1,\ldots,\tau_r,\ 0,\ldots,0), \\
A_3 &= [1+\ \rho_j+\ \sigma_j\,\eta_{g,k}:\ c'_j,\ldots,c_j^{(r)},\ 0,\ldots,1,\ldots,0]_{1,h}, \\
A_4 &= [1+\ \rho_j+\ \sigma_j\,\eta_{g,k}\ :\ c'_j,\ldots,c_j^{(r)},\ 0,\ldots,0]_{1,h},
\end{aligned}
$$

$$A_5 = [1 - a - b - (\lambda + \mu)\,\eta_{g,k} \,:\, (\gamma_1 + \tau_1)\,, ..., (\gamma_r + \tau_r),\, 1, ..., 1]\,;$$

$$R_1 = \begin{cases} z_1(n-m)^{\gamma_1+\tau_1} / \prod_{j=1}^{h} (p_j m + q_j)^{c_j^{'}} \\ \vdots \\ z_r(n-m)^{\gamma_r+\tau_r} / \prod_{j=1}^{h} (p_j m + q_j)^{c_j^{(r)}} \end{cases}$$

$$\text{and } R_2 = \begin{cases} (n-m)p_1/(p_1 m + q_1) \\ \vdots \\ (n-m)p_h/(p_h m + q_h) \end{cases}.$$

Proof. In order to prove (8), expand the multivariable I-function in terms of Mellin-Barnes type of contour integral by (4) and $\bar{H}$-function given by (7) and interchange the order of summation and integration (which is permissible under the conditions of validity stated above). Making use of the results in (3), (5) and (6), we get the desired result.

3 Special Cases

I. If we set $\gamma_1 = 0 = ... = \gamma_r and \lambda = 0$, the integral (8) reduces to

$$\int_m^n (t-m)^{a-1}\,(n-t)^{b-1} \left\{ \prod_{j=1}^{h} (p_j t + q_j)^{\rho_j} \right\}$$
$$.\bar{H}_{P,Q}^{M,N} \left[x(n-t)^{\mu} \prod_{j=1}^{h} (p_j t + q_j)^{\sigma_j} \right]$$
$$.I \left[\begin{array}{c} z_1(n-t)^{\tau_1} \prod_{j=1}^{h} (p_j t + q_j)^{-c_j^{'}} \\ \vdots \\ z_r(n-t)^{\tau_r} \prod_{j=1}^{h} (p_j t + q_j)^{-c_j^{(r)}} \end{array} \right] dt$$

$$= F_1 \sum_{g=1}^{M} \sum_{k=0}^{\infty} \frac{(-1)^k\,\varphi(\eta_{g,k})}{k!F_g}\, x^{\eta_{g,k}}\, F_2$$
$$.I^{0,\,n_2;\,0,n_3\;;...;\,0,\,n_r+h+2\;:\,(m',n')\,;...;\,(m^{(r)},n^{(r)})\,;\,(1,0)\,;...;\,(1,0)}_{p_2,q_2;\,p_3,q_3\,;...;\,p_r+h+2,q_r+h+1:(p',q');...;(p^{(r)},q^{(r)})\,;\,(0,1)\,;...;\,(0,1)}$$
$$\left[\begin{array}{l} B_1,\, B_2,\, B_3,\, (a_{2j};\alpha_{2j}^{'},\alpha_{2j}^{''})_{1,p_2};\,(a_{3j};\alpha_{3j}^{'},\alpha_{3j}^{''},\alpha_{3j}^{'''})_{1,p_3};...;\,(a_{rj};\alpha_{rj}^{'},...,\alpha_{rj}^{(r)},0,...,0)_{1,p_r}; \\ (b_{2j};\beta_{2j}^{'},\beta_{2j}^{''})_{1,q_2};\,(b_{3j};\beta_{3j}^{'},\beta_{3j}^{''},\beta_{3j}^{'''})_{1,q_3};...;\,(b_{rj};\beta_{rj}^{'},...,\beta_{rj}^{(r)},0,...,0)_{1,q_r}, \end{array} \right.$$
$$\left. \begin{array}{l} (a_j^{'},\alpha_j^{'})_{1,p'};...;\,(a_j^{(r)},\alpha_j^{(r)})_{1,p^{(r)}};\,-\;;...;\,-\;;\,P_1 \\ B_4,\,B_5,\,(b_j,\beta_j)_{1,q'};...;\,(b_j^{(r)},\beta_j^{(r)})_{1,q^{(r)}};\,(0,1)\,...\,;\,(0,1)\,;\,P_2 \end{array} \right], \tag{9}$$

which holds true under the same conditions as given in (8).
Where

$$F_1 = (n-m)^{(a+b-1)} \left(\prod_{j=1}^{h} (p_j m + q_j)^{\rho_j} \right),$$

$$F_2 = (n-m)^{\mu\, \eta_{g,k}} \left(\prod_{j=1}^{h} (p_j m + q_j)^{\sigma_j \eta_{g,k}} \right),$$

$$B_1 = [1 - a \ : \ \overbrace{0, ..., 0}^{r},\ 1, ..., 1],$$

$$B_2 = [1 - b - \mu\, \eta_{g,k} \ : \ \tau_1, ..., \tau_r,\ 0, ..., 0],$$

$$B_3 = [1 + \rho_j + \sigma_j\, \eta_{g,k} : \ c_j^{'}, ..., c_j^{(r)},\ 0, ..., 1, ...0]_{1,h},$$

$$B_4 = [1 + \rho_j + \sigma_j\, \eta_{g,k} : \ c_j^{'}, ..., c_j^{(r)},\ 0, ..., 0]_{1,h},$$

$$B_5 = [1 - a - b - \mu\eta_{g,k} \ : \ \tau_1, ..., \tau_r,\ 1, ..., 1],$$

$$P_1 = \begin{cases} z_1 (n-m)^{\tau_1} / \prod_{j=1}^{h} (p_j m + q_j)^{c_j^{'}} \\ \vdots \\ z_r (n-m)^{\tau_r} / \prod_{j=1}^{h} (p_j m + q_j)^{c_j^{(r)}} \end{cases}$$

and

$$\mathrm{P}_2 = \begin{cases} (\mathrm{n}-\mathrm{m})\mathrm{p}_1/(\mathrm{p}_1\mathrm{m} + \mathrm{q}_1) \\ \vdots \\ (\mathrm{n}-\mathrm{m})\mathrm{p}_\mathrm{h}/(\mathrm{p}_\mathrm{h}\mathrm{m} + \mathrm{q}_\mathrm{h}). \end{cases}$$

II. For $\tau_1 = 0 = ... = \tau_\mathrm{r}$ and $\mu = 0$, the integral (8) reduces to

$$\int_\mathrm{m}^\mathrm{n} (\mathrm{t}-\mathrm{m})^{\mathrm{a}-1} (\mathrm{n}-\mathrm{t})^{\mathrm{b}-1} \left\{ \prod_{\mathrm{j}=1}^{\mathrm{h}} (\mathrm{p_j t} + \mathrm{q_j})^{\rho_\mathrm{j}} \right\}$$

$$.\bar{\mathrm{H}}_{\mathrm{P,Q}}^{\mathrm{M,N}} \left[\mathrm{x}(\mathrm{t}-\mathrm{m})^{\lambda} \prod_{\mathrm{j}=1}^{\mathrm{h}} (\mathrm{p_j t} + \mathrm{q_j})^{\sigma_\mathrm{j}} \right]$$

$$.\mathrm{I} \begin{bmatrix} \mathrm{z}_1 (\mathrm{t}-\mathrm{m})^{\gamma_1} \prod_{\mathrm{j}=1}^{\mathrm{h}} (\mathrm{p_j t} + \mathrm{q_j})^{-\mathrm{c}_\mathrm{j}^{'}} \\ \vdots \\ \mathrm{z_r} (\mathrm{t}-\mathrm{m})^{\gamma_\mathrm{r}} \prod_{\mathrm{j}=1}^{\mathrm{h}} (\mathrm{p_j t} + \mathrm{q_j})^{-\mathrm{c}_\mathrm{j}^{(\mathrm{r})}} \end{bmatrix} \mathrm{dt}$$

$$= \Gamma(b)\, L_1 \sum_{g=1}^{M} \sum_{k=0}^{\infty} \frac{(-1)^k \varphi(\eta_{g,k})}{k! F_g} x^{\eta_{g,k}} L_2$$

$$.\mathrm{I}^{0,\,\mathrm{n}_2;\,0,\mathrm{n}_3\ ;...;\,0,\,\mathrm{n_r}+\mathrm{h}+1\ :\,(\mathrm{m}',\mathrm{n}');...;(\mathrm{m}^{(\mathrm{r})},\mathrm{n}^{(\mathrm{r})});\ (1,0);...;(1,0)}_{\mathrm{p}_2,\mathrm{q}_2;\,\mathrm{p}_3,\mathrm{q}_3\,;...;\,\mathrm{p_r}+\mathrm{h}+1,\mathrm{q_r}+\mathrm{h}+1:(\mathrm{p}',\mathrm{q}');...;(\mathrm{p}^{(\mathrm{r})},\mathrm{q}^{(\mathrm{r})});\ (0,1);...;(0,1)}$$

$$\left[\begin{matrix} \mathrm{K}_1, \mathrm{K}_2, (\mathrm{a}_{2\mathrm{j}}; \alpha_{2\mathrm{j}}^{'}, \alpha_{2\mathrm{j}}^{''})_{1,\mathrm{p}_2}; (\mathrm{a}_{3\mathrm{j}}; \alpha_{3\mathrm{j}}^{'}, \alpha_{3\mathrm{j}}^{''}, \alpha_{3\mathrm{j}}^{'''})_{1,\mathrm{p}_3};...;(\mathrm{a_{rj}}; \alpha_{\mathrm{rj}}^{'},...,\alpha_{\mathrm{rj}}^{(\mathrm{r})}, 0,...,0)_{1,\mathrm{p_r}}; \\ (\mathrm{b}_{2\mathrm{j}}; \beta_{2\mathrm{j}}^{'}, \beta_{2\mathrm{j}}^{''})_{1,\mathrm{q}_2}; (\mathrm{b}_{3\mathrm{j}}; \beta_{3\mathrm{j}}^{'}, \beta_{3\mathrm{j}}^{''}, \beta_{3\mathrm{j}}^{'''})_{1,\mathrm{q}_3};...;(\mathrm{b_{rj}}; \beta_{\mathrm{rj}}^{'},...,\beta_{\mathrm{rj}}^{(\mathrm{r})}, 0,...,0)_{1,\mathrm{q_r}}; \end{matrix} \right.$$

$$\left.\begin{matrix} (a_j^{'},\alpha_j^{'})_{1,p'};...;(a_j^{(r)},\alpha_j^{(r)})_{1,p^{(r)}};-\ ;...;-\ ;T_1 \\ K_3,K_4,(b_j,\beta_j)_{1,q'};...;(b_j^{(r)},\beta_j^{(r)})_{1,q^{(r)}};(0,1)...;(0,1);T_2 \end{matrix}\right], \tag{10}$$

which holds under the conditions surrounding equation (8).
Here

$$L_1 = (n-m)^{a+b-1}\left(\prod_{j=1}^{h}(p_j m + q_j)^{\rho_j}\right),$$

$$L_2 = (n-m)^{\lambda\,\eta_{g,k}}\left(\prod_{j=1}^{h}(p_j m + q_j)^{\sigma_j \eta_{g,k}}\right),$$

$$K_1 = [1-a-\lambda\,\eta_{g,k} : \gamma_1,...,\gamma_r, 1,...,1],$$
$$K_2 = [1+\rho_j+\sigma_j\,\eta_{g,k} : c_j^{'},...,c_j^{(r)}, 0,...,1,...0]_{1,h},$$
$$K_3 = [1+\rho_j+\sigma_j\,\eta_{g,k} : c_j^{'},...,c_j^{(r)}, 0,...,0]_{1,h},$$

$$K_4 = [1-a-b-\eta_{g,k} : \gamma_1,...,\gamma_r, 1,...,1],$$

$$T_1 = \begin{cases} z_1(n-m)^{\gamma_1}/\prod_{j=1}^{h}(p_j m + q_j)^{c_j^{'}} \\ \vdots \\ z_r(n-m)^{\gamma_r}/\prod_{j=1}^{h}(p_j m + q_j)^{c_j^{(r)}}, \end{cases}$$

$$T_2 = \begin{cases} (n-m)p_1/(p_1 m + q_1) \\ \vdots \\ (n-m)p_h/(p_h m + q_h) \end{cases}$$

III. When $\tau_1 = ... = \tau_r = 0 = \gamma_1 = ... = \gamma_r = 0and\lambda = 0 = \mu$, integral (8) reduces to

$$\int_m^n (t-m)^{a-1}(n-t)^{b-1}\left\{\prod_{j=1}^{h}(p_j t + q_j)^{\rho_j}\right\}$$

$$.\bar{H}_{P,Q}^{M,N}\left[x\prod_{j=1}^{h}(p_j t + q_j)^{\sigma_j}\right] I\left[\begin{matrix} z_1\prod_{j=1}^{h}(p_j t + q_j)^{-c_j^{'}} \\ \vdots \\ z_r\prod_{j=1}^{h}(p_j t + q_j)^{-c_j^{(r)}} \end{matrix}\right] dt$$

$$= \Gamma(b)\,E_1\sum_{g=1}^{M}\sum_{k=0}^{\infty}\frac{(-1)^k\varphi(\eta_{g,k})}{k!F_g}x^{\eta_{g,k}}E_2$$

$$.I^{0,n_2;0,n_3;...;0,n_r+h+1:(m',n');...;(m^{(r)},n^{(r)});(1,0);...;(1,0)}_{p_2,q_2;p_3,q_3;...;p_r+h+1,q_r+h+1:(p',q');...;(p^{(r)},q^{(r)});(0,1);...;(0,1)}$$

$$\left[\begin{array}{l} Q_1, Q_2, (a_{2j}; \alpha_{2j}^{'}, \alpha_{2j}^{''})_{1,p_2}; (a_{3j}; \alpha_{3j}^{'}, \alpha_{3j}^{''}, \alpha_{3j}^{'''})_{1,p_3}; ...; (a_{rj}; \alpha_{rj}^{'}, ..., \alpha_{rj}^{(r)}, 0, ..., 0)_{1,p_r}; \\ (b_{2j}; \beta_{2j}^{'}, \beta_{2j}^{''})_{1,q_2}; (b_{3j}; \beta_{3j}^{'}, \beta_{3j}^{''}, \beta_{3j}^{'''})_{1,q_3}; ...; (b_{rj}; \beta_{rj}^{'}, ..., \beta_{rj}^{(r)}, 0, ..., 0)_{1,q_r}; \end{array}\right. \tag{11}$$

$$\left.\begin{array}{l} (a_j^{'}, \alpha_j^{'})_{1,p'}; ...; (a_j^{(r)}, \alpha_j^{(r)}); -; ...; -; U_1 \\ Q_3, Q_4, (b_j, \beta_j)_{1,q'}; ...; (b_j^{(r)}, \beta_j^{(r)}); (0,1) ...; (0,1); U_2 \end{array}\right],$$

valid under the same conditions as required in (8).
Where

$$E_1 = (n-m)^{a+b-1}\left(\prod_{j=1}^{h}(p_j m + q_j)^{\rho_j}\right),$$

$$E_2 = \prod_{j=1}^{h}(p_j m + q_j)^{\sigma_j \eta_{g,k}};$$

$$Q_1 = [1 - a :, \overbrace{0, ..., 0}^{r}, 1, ..., 1],$$

$$Q_2 = [1 + \rho_j + \sigma_j\, \eta_{g,k} : c_j^{'}, ..., c_j^{(r)}, 0, ..., 1, ...0]_{1,h},$$

$$Q_3 = [1 + \rho_j + \sigma_j\, \eta_{g,k} : c_j^{'}, ..., c_j^{(r)}, 0, ..., 0]_{1,h}, \quad Q_4 = [1 - a - b : \overbrace{0, ..., 0}^{r}, 1, ..., 1],$$

$$U_1 = \begin{cases} \prod_{j=1}^{h}(p_j m + q_j)^{-c_j^{'}} \\ \vdots \\ \prod_{j=1}^{h}(p_j m + q_j)^{-c_j^{(r)}}, \end{cases}$$

$$U_2 = \begin{cases} (n-m)p_1/(p_1 m + q_1) \\ \vdots \\ (n-m)p_h/(p_h m + q_h). \end{cases}$$

4 Main Theorem

Let

$$f(t) = (t-m)^{a-1}\left\{\prod_{j=1}^{h}(p_j t + q_j)^{\rho_j}\right\}$$

$$.\bar{H}_{P,Q}^{M,N}\left[X(t-m)^{\lambda}\prod_{j=1}^{h}(p_j t + q_j)^{\sigma_j}\right]$$

$$.I\left[\begin{array}{c} W_1(t-m)^{\gamma_1}\prod_{j=1}^{h}(p_j t + q_j)^{-c_j^{'}} \\ \\ W_r(t-m)^{\gamma_r}\prod_{j=1}^{h}(p_j t + q_j)^{-c_j^{(r)}} \end{array}\right],$$

then

$$
\begin{aligned}
{}_{m}D_{y}^{-b}\left[f(y)\right] &= I_1 \sum_{g=1}^{M} \sum_{k=0}^{\infty} \frac{(-1)^k \varphi(\eta_{g,k})}{k! F_g} X^{\eta_{g,k}} I_2 \\
&.I^{0,n_2;0,n_3;\ldots;0,n_r+h+1:(m',n');\ldots;(m^{(r)},n^{(r)});(1,0);\ldots;(1,0)}_{p_2,q_2;p_3,q_3;\ldots;p_r+h+1,q_r+h+1:(p',q');\ldots;(p^{(r)},q^{(r)});(0,1);\ldots;(0,1)} \\
&\left[\begin{matrix} D_1, D_2, (a_{2j};\alpha'_{2j},\alpha''_{2j})_{1,p_2}; (a_{3j};\alpha'_{3j},\alpha''_{3j},\alpha'''_{3j})_{1,p_3};\ldots;(a_{rj};\alpha'_{rj},\ldots,\alpha^{(r)}_{rj},0,\ldots,0)_{1,p_r}; \\ (b_{2j};\beta'_{2j},\beta''_{2j})_{1,q_2}; (b_{3j};\beta'_{2j},\beta''_{3j},\beta'''_{3j})_{1,q_3};\ldots;(b_{rj};\beta'_{rj},\ldots,\beta^{(r)}_{rj},0,\ldots,0)_{1,q_r}; \end{matrix}\right. \\
&\left.\begin{matrix} (a'_j,\alpha'_j)_{1,p'};\ldots;(a^{(r)}_j,\alpha^{(r)}_j)_{1,p^{(r)}}; -;\ldots; -; Y_1 \\ D_3, D_4, (b_j,\beta_j)_{1,q'};\ldots;(b^{(r)}_j,\beta^{(r)}_j)_{1,q^{(r)}};(0,1)\ldots;(0,1);Y_2 \end{matrix}\right]
\end{aligned}
\qquad (12)
$$

valid under the same conditions as needed for integral (8).
Where

$$
\begin{aligned}
I_1 &= (y-m)^{a+b-1}\left(\prod_{j=1}^{h}(p_j m + q_j)^{\rho_j}\right), \\
I_2 &= (y-m)^{\lambda\,\eta_{g,k}}\left(\prod_{j=1}^{h}(p_j m + q_j)^{\sigma_j \eta_{g,k}}\right), \\
D_1 &= [1-a-\lambda\,\eta_{g,k} : \gamma_1,\ldots,\gamma_r, 1,\ldots,1], \\
D_2 &= [1+\rho_j+\sigma_j\,\eta_{g,k} : c'_j,\ldots,c^{(r)}_j, 0,\ldots,1,\ldots 0]_{1,h}, \\
D_3 &= [1+\rho_j+\sigma_j\,\eta_{g,k} : c'_j,\ldots,c^{(r)}_j, 0,\ldots,0]_{1,h}, \\
D_4 &= [1-a-b-\lambda\eta_{g,k} : \gamma_1,\ldots,\gamma_r, 1,\ldots,1], \\
Y_1 &= \begin{cases} W_1(y-m)^{\gamma_1} / \prod_{j=1}^{h}(p_j m+q_j)^{c'_j} \\ \vdots \\ W_r(y-m)^{\gamma_r} / \prod_{j=1}^{h}(p_j m+q_j)^{c^{(r)}_j} \end{cases}
\end{aligned}
$$

and

$$
Y_2 = \begin{cases} (y-m)p_1/(p_1 m + q_1) \\ \vdots \\ (y-m)p_h/(p_h m + q_h). \end{cases}
$$

5 Interesting Special Cases

I. For $\gamma_1 = \ldots = \gamma_r = 0$ and $\lambda = 0$, integral (12) reduces to

$$
{}_{m}D_{y}^{-b}\left[f(y)\right] = I_1 \sum_{g=1}^{M} \sum_{k=0}^{\infty} \frac{(-1)^k \varphi(\eta_{g,k})}{k! F_g} X^{\eta_{g,k}} I_2
$$

$$
.I^{0,n_2;0,n_3;\ldots;0,n_r+h+1:(m',n');\ldots;(m^{(r)},n^{(r)});(1,0);\ldots;(1,0)}_{p_2,q_2;p_3,q_3;\ldots;p_r+h+1,q_r+h+1:(p',q');\ldots;(p^{(r)},q^{(r)});(0,1);\ldots;(0,1)}
\left[\begin{matrix} D_1, D_2, (a_{2j};\alpha'_{2j},\alpha''_{2j})_{1,p_2};(a_{3j};\alpha'_{3j},\alpha''_{3j},\alpha'''_{3j})_{1,p_3};\ldots;(a_{rj};\alpha'_{rj},\ldots,\alpha^{(r)}_{rj},0,\ldots,0)_{1,p_r}; \\ (b_{2j};\beta'_{2j},\beta''_{2j})_{1,q_2};(b_{3j};\beta'_{3j},\beta''_{3j},\beta'''_{3j})_{1,q_3};\ldots;(b_{rj};\beta'_{rj},\ldots,\beta^{(r)}_{rj},0,\ldots,0)_{1,q_r}; \end{matrix}\right.
$$

$$
\left.\begin{matrix} (a'_j,\alpha'_j)_{1,p'};\ldots;(a^{(r)}_j,\alpha^{(r)}_j)_{1,p^{(r)}};-;\ldots;-;Y_1 \\ D_3, D_4, (b_j,\beta_j)_{1,q'};\ldots;(b^{(r)}_j,\beta^{(r)}_j)_{1,q^{(r)}};(0,1);\ldots;(0,1);Y_2 \end{matrix}\right],
$$

which holds true under the same conditions as given in (8) and where $I_1, I_2, D_1, D_2, D_3, D_4, Y_1$ *and* Y_2 are the same as in integral (12) after eliminating λ and γ_i $(i = 1, \ldots, r)$.

II. If we set $n_2 = n_3 = \ldots = n_{r-1} = 0 = p_2 = p_3 = \ldots = p_{r-1} = q_1 = q_2 = \ldots = q_{r-1}$ and

$$
\lambda = 0,\ \mu = 0,\ \sigma_j \to 0,
$$

the results given in (8) and (12) reduces to the known results obtained by Saigo and Saxena [6].

III. For $n_2 = n_3 = \ldots = n_{r-1} = 0 = p_2 = \ldots = p_{r-1} = q_1 = \ldots = q_{r-1}$ and $\lambda = 0$, $\mu = 0$, $\sigma_j \to 0$, $\gamma_i = 0$ $(i = 1, \ldots, r)$ and $h = 1$ in (12), then we arrive at the results given by Srivastava and Hussain [11].

IV. On specializing the parameters, we get the results recently obtained by Chaurasia and Singhal [1].

ACKNOWLEDGEMENT

The authors are grateful to Professor H.M. Srivastava, University of Victoria, Canada for his kind help and valuable suggestions in the preparation of this paper.

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Received: November, 2009