Global Asymptotic Stability of a Higher Order Difference Equation

M.A. Al-Shabi

Department of Computer Science
College of Computer,
Qassim University, Buraidah,
51411, Saudi Arabia.

R. Abo-Zeid

Department of Basic Science
faculty of Engineering
October 6 university
6th of October Governorate, Egypt
abuzead73@yahoo.com

Abstract

The aim of this work is to investigate the global stability, periodic nature, oscillation and boundedness of the positive solutions of the difference equation

\[ x_{n+1} = \frac{Ax_{n-2r-1}}{B + Cx_{n-2l}x_{n-2k}}, \quad n = 0, 1, 2, \ldots \]

where \( A, B, C \) are nonnegative real numbers and \( l, r, k \) are nonnegative integers, such that \( l \leq k \) and \( r \leq k \).

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1 Introduction

Difference equations have always played an important role in the construction and analysis of mathematical models of biology, ecology, physics, economic processes, etc. [6].
Recently there has been a great interest in studying the qualitative properties of rational difference equations. For the systematical studies of rational and nonrational difference equations, one can refer to the monographs [7, 4, 11, 5, 6] and the papers [2, 3, 12, 13, 14, 15, 10, 9, 8] and references therein.

The study of nonlinear rational difference equations of higher order is of paramount importance, since we still know so little about such equations. In [1], we have discussed the asymptotic behavior of solutions of the difference equation

\[ x_{n+1} = \frac{Ax_{n-1}}{B + C \prod_{i=l}^{k} x_{n-2i}}, \quad n = 0, 1, 2, \ldots \]

where \( A, B, C \) are nonnegative real numbers and \( l, k \) are nonnegative integers, such that \( l \leq k \).

In this paper, we study the global asymptotic stability of the difference equation

\[ x_{n+1} = \frac{Ax_{n-2r-1}}{B + C x_{n-2l} x_{n-2k}}, \quad n = 0, 1, 2, \ldots \] (1)

where \( A, B, C \) are nonnegative real numbers and \( l, r, k \) are nonnegative integers, such that \( l \leq k \) and \( r \leq k \).

The following particular cases can be obtained:

1. When \( A = 0 \), equation (1) reduces to \( x_{n+1} = 0, n = 0, 1, 2, \ldots \) which has the trivial solution.

2. When \( B = 0 \), equation (1) reduces to

\[ x_{n+1} = \frac{Ax_{n-2r-1}}{C x_{n-2l} x_{n-2k}}, \quad n = 0, 1, 2, \ldots \]

This equation can be reduced to the linear difference equation

\[ y_{n+1} - y_{n-2r-1} + y_{n-2l} + y_{n-2k} = \gamma, \]

by taking

\[ x_n = e^{y_n}, \gamma = \ln \frac{A}{C}. \]

3. When \( C = 0 \), equation (1) reduces to \( x_{n+1} = \frac{A}{B} x_{n-2r-1}, n = 0, 1, 2, \ldots \) Which is a linear difference equation.

For various values of \( l, r \) and \( k \), we can get more equations.
2 Preliminaries

Consider the difference equation

\[ x_{n+1} = f(x_n, x_{n-1}, \ldots, x_{n-k}), \quad n = 0, 1, \ldots \]  

(2)

where \( f : \mathbb{R}^{k+1} \to \mathbb{R} \).

Definition 2.1 \cite{11}

An equilibrium point for equation (2) is a point \( \bar{x} \in \mathbb{R} \) such that \( \bar{x} = f(\bar{x}, \bar{x}, \ldots, \bar{x}) \).

Definition 2.2 \cite{11}

1. An equilibrium point \( \bar{x} \) for equation (2) is called locally stable if for every \( \epsilon > 0, \exists \delta > 0 \) such that every solution \( \{x_n\} \) with initial conditions \( x_{-k}, x_{-k+1}, \ldots, x_0 \in [\bar{x} - \delta, \bar{x} + \delta] \) is such that \( x_n \in [\bar{x} - \epsilon, \bar{x} + \epsilon], \forall n \in \mathbb{N} \). Otherwise \( \bar{x} \) is said to be unstable.

2. The equilibrium point \( \bar{x} \) of equation (2) is called locally asymptotically stable if it is locally stable and there exists \( \gamma > 0 \) such that for any initial conditions \( x_{-k}, x_{-k+1}, \ldots, x_0 \in [\bar{x} - \gamma, \bar{x} + \gamma] \), the corresponding solution \( \{x_n\} \) tends to \( \bar{x} \).

3. An equilibrium point \( \bar{x} \) for equation (2) is called global attractor if every solution \( \{x_n\} \) converges to \( \bar{x} \) as \( n \to \infty \).

4. The equilibrium point \( \bar{x} \) for equation (2) is called globally asymptotically stable if it is locally asymptotically stable and global attractor.

The linearized equation associated with equation (2) is

\[ y_{n+1} = \sum_{i=0}^{k} \frac{\partial f}{\partial x_{n-i}}(\bar{x}, \ldots, \bar{x})y_{n-i}, \quad n = 0, 1, 2, \ldots \]  

(3)

the characteristic equation associated with equation (3) is

\[ \lambda^{k+1} - \sum_{i=0}^{k} \frac{\partial f}{\partial x_{n-i}}(\bar{x}, \ldots, \bar{x})\lambda^{k-i} = 0 \]  

(4)

**Theorem 2.1** \cite{11} Assume that \( f \) is a \( C^1 \) function and let \( \bar{x} \) be an equilibrium point of equation (2). Then the following statements are true:

1. If all roots of equation (4) lie in the open disk \( |\lambda| < 1 \), then \( \bar{x} \) is locally asymptotically stable.

2. If at least one root of equation (4) has absolute value greater than one, then \( \bar{x} \) is unstable.
3 Linearized stability analysis

Consider the difference equation

\[ x_{n+1} = \frac{Ax_{n-2r-1}}{B + Cx_{n-2l}x_{n-2k}}, \quad n = 0, 1, 2, \ldots \]

where \( A, B, C \) are nonnegative real numbers and \( l, r, k \) are nonnegative integers, such that \( l \leq k \) and \( r \leq k \).

The change of variables \( x_n = \sqrt{A}y_n \) reduces equation (1) to the difference equation

\[ y_{n+1} = \frac{y_{n-2r-1}}{\gamma + y_{n-2t}y_{n-2k}}, \quad n = 0, 1, 2, \ldots \]  \hspace{1cm} (5)

where \( \gamma = \frac{B}{A} \).

Now we examine the equilibrium points of equation (5) and their local asymptotic behavior. Clearly equation (5) has two nonnegative equilibrium points \( \bar{y} = 0 \) and \( \bar{y} = \sqrt{1 - \gamma} \) when \( \gamma < 1 \) and \( \bar{y} = 0 \) only when \( \gamma \geq 1 \).

The linearized equation associated with equation (5) about \( \bar{y} \) is

\[ z_{n+1} - \frac{1}{\gamma + \bar{y}^2}z_{n-2r-1} + \frac{\bar{y}^2}{(\gamma + \bar{y}^2)^2}(z_{n-2t} + z_{n-2k}) = 0, \quad n = 0, 1, 2, \ldots \]  \hspace{1cm} (6)

the characteristic equation associated with this equation is

\[ \lambda^{2k+1} - \frac{1}{\gamma + \bar{y}^2}\lambda^{2k-2r-1} + \frac{\bar{y}^2}{(\gamma + \bar{y}^2)^2}(\lambda^{2k-2t} + 1) = 0. \]  \hspace{1cm} (7)

We summarize the results of this section in the following theorem.

**Theorem 3.1** 1. If \( \gamma > 1 \), then the zero equilibrium point is locally asymptotically stable.

2. If \( \gamma < 1 \), then the equilibrium points \( \bar{y} = 0 \) and \( \bar{y} = \sqrt{1 - \gamma} \) are unstable (saddle points).

**Proof**

The linearized equation associated with equation (5) about \( \bar{y} = 0 \) is

\[ z_{n+1} - \frac{1}{\gamma}z_{n-2r-1} = 0, \quad n = 0, 1, 2, \ldots \]

The characteristic equation associated with this equation is

\[ \lambda^{2k+1} - \frac{1}{\gamma}\lambda^{2k-2r-1} = 0 \]

so \( \lambda = 0, \lambda = \pm \frac{\sqrt{k}}{\sqrt{\gamma}} \).
1. If $\gamma > 1$, then $|\lambda| < 1$ for all roots and $\bar{y} = 0$ is locally asymptotically stable.

2. If $\gamma < 1$, it follows that $\bar{y} = 0$ is unstable (saddle point). The linearized equation (6) about $\bar{y} = \sqrt{1 - \gamma}$ becomes

\[ z_{n+1} - z_{n-2r-1} + (1 - \gamma)(z_{n-2l} + z_{n-2k}) = 0, \quad n = 0, 1, 2, \ldots \]

The associated characteristic equation is

\[ \lambda^{2k+1} - \lambda^{2k-2r-1} + (1 - \gamma)(\lambda^{2k-2l} + 1) = 0. \]

Let $f(\lambda) = \lambda^{2k+1} - \lambda^{2k-2r-1} + (1 - \gamma)(\lambda^{2k-2l} + 1)$. We can see that $f(\lambda)$ has a real root in $(-\infty, -1)$. Then the point $\bar{y} = \sqrt{1 - \gamma}$ is unstable (saddle point).

4 Global behavior of equation (5)

**Theorem 4.1** If $\gamma > 1$, then the zero equilibrium point is globally asymptotically stable.

**Proof**

Let $\{y_n\}_{n=-2k-1}^\infty$ be a solution of equation (5). Hence

\[ y_{n+1} = \frac{y_{n-2r-1}}{\gamma + y_{n-2l}y_{n-2k}} < \frac{y_{n-2r-1}}{\gamma}, \quad n = 0, 1, 2, \ldots \]

then

\[ y_{2n(r+1)+i} < \frac{1}{\gamma^{n+1}}y_{2n-2r-2}, \quad i = 1, 2, \ldots 2r + 2. \]

Hence each of the subsequences $\{y_{2n(r+1)+i}\}_{n=0}^\infty, i = 1, 2, \ldots, 2r+2$, converges to zero. Therefore

\[ \lim_{n \to \infty} y_n = 0. \]

In view of theorem (3.1), $\bar{y} = 0$ is globally asymptotically stable.

5 Semicycle analysis

**Theorem 5.1** Let $\{y_n\}_{n=-2k}^\infty$ be a nontrivial solution of equation (5) and let $\bar{y}$ denote the unique positive equilibrium of equation (5) such that either,

(C1) $y_{-2k}, y_{-2k+2}, \ldots, y_0 > \bar{y}$ and $y_{-2k+1}, y_{-2k+3}, \ldots, y_{-1} < \bar{y}$

Or

(C2) $y_{-2k}, y_{-2k+2}, \ldots, y_0 < \bar{y}$ and $y_{-2k+1}, y_{-2k+3}, \ldots, y_{-1} > \bar{y}$

is satisfied, then $\{y_n\}_{n=-2k}^\infty$ oscillates about $\bar{y}$ with semicycles of length one.
Proof
Assume that condition \((C_1)\) is satisfied. Then we have
\[
y_1 = \frac{y_{n-2l-1}}{\gamma + y_{n-2l-1}} < \frac{\bar{y}}{\gamma + y^2} = \bar{y},
\]
\[
y_2 = \frac{y_{n-2l+1}}{\gamma + y_{n-2l+1}} > \frac{\bar{y}}{\gamma + y^2} = \bar{y}, \text{ by induction we obtain the result.}
\]
Assume that condition \((C_2)\) is satisfied. Then we have
\[
y_1 = \frac{y_{n-2l-1}}{\gamma + y_{n-2l-1}} > \frac{\bar{y}}{\gamma + y^2} = \bar{y},
\]
\[
y_2 = \frac{y_{n-2l+1}}{\gamma + y_{n-2l+1}} < \frac{\bar{y}}{\gamma + y^2} = \bar{y}, \text{ by induction we obtain the result.}
\]

6. **Case** \(r = k\)

When \(r = k\), equation (5) becomes
\[
y_{n+1} = \frac{y_{n-2k-1}}{\gamma + y_{n-2l} y_{n-2k}} , n = 0, 1, 2, \ldots \tag{8}
\]

The following theorem summarizes the linearized stability analysis of equation (8).

**Theorem 6.1**
1. If \(\gamma > 1\), then the zero equilibrium point is locally asymptotically stable.

2. If \(\gamma < 1\), then the equilibrium points \(\bar{y} = 0\) is unstable (repeller) and \(\bar{y} = \sqrt{1 - \gamma}\) are unstable (saddle points).

**Proof**
It is sufficient to consider the linearized equation
\[
z_{n+1} + \frac{\bar{y}^2}{\gamma + \bar{y}^2} (z_{n-2l} + z_{n-2k}) - \frac{1}{\gamma + \bar{y}^2} z_{n-2k-1} = 0 , n = 0, 1, 2, \ldots
\]
and its associated characteristic equation
\[
\lambda^{2k+2} + \frac{\bar{y}^2}{\gamma + \bar{y}^2} (\lambda^{2k-2l+1} + \lambda) - \frac{1}{\gamma + \bar{y}^2} = 0.
\]

Therefore, the results follows.

**Theorem 6.2** The following statements are true:

1. Assume that \(\gamma > 1\). Then the zero equilibrium point is globally asymptotically stable.

2. Assume that \(\gamma = 1\). Then every solution of equation (8) converges to a periodic solution of equation (8) with period \(2(k+1)\) and there exist periodic solutions of equation (8) with prime period \(2(k+1)\).
3. Assume that $\gamma < 1$. Then there exist solutions of equation (8) which are neither bounded nor persist.

Proof

1. The proof is similar to that of theorem (4.1).

2. Assume that $\gamma = 1$. Let $\{y_n\}_{n=-2k-1}^\infty$ be a solution of equation (8). For $n \geq 0$ we have

$$y_{n+1} = \frac{y_{n-2k-1}}{1+y_{n-2}y_{n-2k}} \leq y_{n-2k-1}, \quad n = 0, 1, 2, \ldots$$

Hence the subsequences $\{y_{2n(k+1)+i}\}_{n=-1}^\infty$ are decreasing for each $1 \leq i \leq 2k+2$. Let

$$\lim_{n \to \infty} y_{(2k+2)n+i} = \rho_i, \quad i = 1, 2, \ldots, 2k+2.$$ 

It is clear that $\{\ldots, \rho_1, \rho_2, \ldots, \rho_{2k+2}, \rho_1, \rho_2, \ldots, \rho_{2k+2}, \ldots\}$ is a $2(k+1)$-periodic solution of equation (8).

Now let $\varphi_0, \varphi_1, \ldots, \varphi_k$ be distinct positive real numbers. It follows that the sequence

$$\ldots, \varphi_0, 0, \varphi_1, 0, \ldots, \varphi_k, 0, \varphi_0, 0, \varphi_1, 0, \ldots, \varphi_k, \ldots$$

is a periodic solution of equation (8) with prime period $2(k+1)$.

3. Assume that $\gamma < 1$. Let $\{y_n\}_{n=-2k-1}^\infty$ be a nontrivial solution of equation (8) and let $\bar{y}$ denote the unique positive equilibrium of equation (8) such that

$0 < \bar{y} < y_{-2k}, y_{-2k+2}, \ldots, y_0$ and $0 < y_{-2k-1}, y_{-2k+1}, y_{-2k+3}, \ldots, y_{-1} < \bar{y}$

is satisfied.

It follows that for all $m \geq 0$ and $0 \leq j \leq k$, we have

$$y_{(2k+2)(m+1)+2j} > y_{(2k+2)m+2j}$$

and

$$y_{(2k+2)(m+1)+2j+1} < y_{(2k+2)m+2j+1}$$

Hence for each $0 \leq j \leq k$

$$\lim_{m \to \infty} y_{(2k+2)m+2j} = L_{2j} \in (\sqrt{1 - \gamma}, \infty) \quad \text{and} \quad \lim_{m \to \infty} y_{(2k+2)m+2j+1} = L_{2j+1} \in [0, \sqrt{1 - \gamma}).$$

We show that for each $0 \leq j \leq k$, $L_{2j+1} = 0$.

For the sake of contradiction, suppose that there exists $j \in \{0, 1, \ldots, k\}$
with $L_{2j+1} \in (0, \sqrt{1-\gamma})$.

Then

$$L_{2j+1} = \lim_{m \to \infty} y(2k+2)(m+1)+2j+1$$

$$= \lim_{m \to \infty} \frac{y(2k+2)m+2j+1}{\gamma + y(2k+2)(m+1)+2j-2\gamma y(2k+2)m+2j+2}$$

$$= \frac{L_{2j+1}}{\gamma + L_{2j-2}L_{2j+2}}.$$ 

So as

$$\lim_{m \to \infty} y(2k+2)m+2j+1 = L_{2j+1} \in (0, \sqrt{1-\gamma})$$

we have

$$1 = \gamma + L_{2j-2}L_{2j+2} > 1$$

which is a contradiction.

Thus it is true that for each $0 \leq j \leq k$, $L_{2j+1} = 0$, and so

$$\lim_{n \to \infty} y_{2n+1} = 0.$$ 

Now we show that for each $0 \leq j \leq k$, $L_{2j} = \infty$.

For the sake of contradiction, suppose that there exists $j \in \{0, 1, \ldots, k\}$ with $L_{2j} \in (\sqrt{1-\gamma}, \infty)$.

Then

$$L_{2j} = \lim_{m \to \infty} y(2k+2)(m+1)+2j$$

$$= \lim_{m \to \infty} \frac{y(2k+2)m+2j}{\gamma + y(2k+2)(m+1)+2j-2\gamma y(2k+2)m+2j+1}$$

$$= \frac{L_{2j}}{\gamma}.$$ 

So $\gamma = 1$, which is a contradiction. Hence $\lim_{n \to \infty} y_{2n} = \infty$, and the proof is complete.

References


Global asymptotic stability


[8] S. Stević, On the recursive sequence $x_{n+1} = \frac{g(x_n, x_{n-1})}{A + x_n}$, Appl. Math. Lett. 15 (2002) 305-308.


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