

## Study of Chaos Synchronization

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### Abstract

In this paper, we define a new criterion to identify a dynamical system and apply control laws in a simpler manner. We study chaos synchronization by using linear feedback control laws for generalized nonlinear dynamical systems exhibiting a chaotic attractor for different parameter values. We have constructed a theorem for synchronization of an  $n$ -dimensional generalized nonlinear dynamical system. Numerical simulations are used to verify the theoretical results. An excellent agreement has been achieved between the theoretical and computational results.

**Keywords:** Generalized dynamical systems, Lyapunov function, Positive-definite matrices, Negative-definite matrices

## 1. Introduction

Chaos synchronization is a phenomenon that may appear when either two, or more, chaotic oscillators are coupled, or a chaotic oscillator drives another chaotic oscillator. Because of the butterfly effect, which causing the exponential divergence of the trajectories of two identical chaotic system started with nearly the same initial conditions, having two chaotic system evolving in synchrony might appear quite surprising. However, synchronization of coupled or driven chaotic oscillators is a phenomenon that a well established experimentally as well as theoretically. It has been received that chaos synchronization is quite a rich phenomenon that may present a variety of forms. The evolution of two chaotic oscillators may appear as: Identical synchronization, generalized synchronization, Phase synchronization, anticipated lag synchronization and Amplitude envelope synchronization. All these forms of synchronization share the property of asymptotic stability consequently once the synchronized state is reached, the effect of a small perturbation destroying synchronization is rapidly damped, and synchronization is recovered again. Mathematically, asymptotic stability is characterized by a positive Lyapunov exponent of the system comprised of the two oscillators. When the chaotic synchronization is achieved the Lyapunov exponent becomes negative.

In 1963, Lorenz simplified the Navier-Stokes equations to modeling weather forecasting and discovered sensitivity dependence on initial conditions in a set of three ordinary differential equations (Li, T.Y., Yorke, J.A., 1975) first presented the name “chaos” in the sense of “period three implies chaos”. Chaos embodies three important principles: extreme sensitivity to initial conditions, cause and effect being not proportional, and the nonlinearity. After the discovery of Lorenz system, some more chaotic systems have been constructed such as Rossler system, hyperchaotic Rossler system, Chua’s circuit, Henon attractor, logistic map, Chen system, generalized Lorenz system etc.

In this paper, we focus to study the chaos synchronization of generalized nonlinear dynamical systems which can exhibit a chaotic attractor for different parameter values by using linear feedback control laws. We propose a theorem for synchronization of an n-dimensional generalized nonlinear dynamical system. Numerical simulations are used to verify the theoretical results.

## 2. Formulation of nonlinear dynamical systems

To study synchronization of two nonlinear dynamical systems we use the following notations and terminology-

$$\mathbb{R}_+ = [0, +\infty)$$

$$J = [t_0, +\infty) \quad t_0 \geq 0$$

$$\mathbb{R}^n = \text{n-dimensional Euclidean space.}$$

$$\|x\| = \left( \sum_{i=1}^n x_i^2 \right)^{1/2}; \quad x = (x_1, x_2, \dots, x_n) \in \mathbb{R}^n.$$

$A = [a_{ij}]_{n \times n}$  matrix

$\lambda_{\max}$  = maximal eigen value of the matrix A.

$\lambda_{\min}$  = minimal eigen value of the matrix A.

I = The identity matrix of order m.

In order to find a sufficient synchronization criterion the following assumption on the drive system is needed. This assumption is in the light of the dynamical system being free and chaotic and based on a well-known fact that chaotic attractors are bounded in phase space

**Assumption**

For any bounded initial state  $x_0$  within the defined domain of the drive system, there exist some finite real constants  $\underline{M}_i(x_0)$  and  $\overline{M}_i(x_0)$  such that

$$\underline{M}_i \leq x_i(t, x_0) \leq \overline{M}_i, \quad i = 1, 2, 3, \quad \forall t \geq 0.$$

Keeping in view these facts we obtain the following results which seem to be very significant to develop the subject of chaos theory.

Consider the drive system:

$$\dot{x}_d[t] = F(t, x_d), \tag{2.1}$$

and response system:

$$\dot{y}_r[t] = F(t, y_r) + u[t] \tag{2.2}$$

where the subscripts “d” and “r” stand for the drive and response systems respectively. If we denote

$x_d = (x_{1d}, x_{2d}, x_{3d}, \dots, x_{kd}, \dots, x_{nd})^T$ ,  $y_r = (y_{1r}, y_{2r}, y_{3r}, \dots, y_{kr}, \dots, y_{nr})^T$  as drive and response system variables respectively and

$$F(t, x_d) = Ax_d + f(t, x_d) \tag{2.3}$$

$$F(t, y_r) = Ay_r + f(t, y_r) \tag{2.4}$$

where  $F: \mathbb{R}_+ \times \mathbb{R}^n \rightarrow \mathbb{R}^n$  is a function that consists of linear and non linear functions  $Ax_d$ ,  $Ay_r$  and  $f(t, x_d)$ ,  $f(t, y_r)$  respectively.

$u[t] = (u_1[t], u_2[t], u_3[t], \dots, u_k[t], \dots, u_n[t])^T$  is a control function added such that response system remain bounded after time t. On the basis of all explained terms, we can restate (2.1) and (2.2) as drive and response systems respectively as follows:

$$\begin{aligned} \dot{x}_{1d}[t] &= a_{11}x_{1d}[t] + \dots + a_{1k}x_{kd}[t] + \dots + a_{1n}x_{nd}[t] + f_1(t, x_d) \\ \dot{x}_{2d}[t] &= a_{21}x_{1d}[t] + \dots + a_{2k}x_{kd}[t] + \dots + a_{2n}x_{nd}[t] + f_2(t, x_d) \\ &\vdots \\ \dot{x}_{kd}[t] &= a_{k1}x_{1d}[t] + \dots + a_{kk}x_{kd}[t] + \dots + a_{kn}x_{nd}[t] + f_k(t, x_d) \\ &\vdots \\ \dot{x}_{nd}[t] &= a_{n1}x_{1d}[t] + \dots + a_{nk}x_{kd}[t] + \dots + a_{nn}x_{nd}[t] + f_n(t, x_d) \end{aligned} \tag{2.5}$$

$$\begin{aligned}
 \dot{y}_{1r}[t] &= a_{11}y_{1r}[t] + \dots + a_{1k}y_{kr}[t] + \dots + a_{1n}y_{nr}[t] + f_1(t, y_r) + u_1[t] \\
 \dot{y}_{2r}[t] &= a_{21}y_{1r}[t] + \dots + a_{2k}y_{kr}[t] + \dots + a_{2n}y_{nr}[t] + f_2(t, y_r) + u_2[t] \\
 &\vdots \\
 \dot{y}_k[t] &= a_{k1}y_{1r}[t] + \dots + a_{kk}y_{kr}[t] + \dots + a_{kn}y_{nr}[t] + f_k(t, y_r) + u_k[t] \\
 &\vdots \\
 \dot{y}_{nr}[t] &= a_{n1}y_{1r}[t] + \dots + a_{nk}y_{kr}[t] + \dots + a_{nn}y_{nr}[t] + f_n(t, y_r) + u_n[t]
 \end{aligned}
 \tag{2.6}$$

Let the error state be

$$\begin{aligned}
 e[t] &= (e_1[t], e_2[t], e_3[t], \dots, e_1[t], \dots, e_n[t])^T \\
 &= (x_{1d}[t] - y_{1r}[t], \dots, x_{kd}[t] - y_{kr}[t], \dots, x_{nd}[t] - y_{nr}[t])^T
 \end{aligned}
 \tag{2.7}$$

Then the error dynamical system on e[t] given by-

$$\begin{aligned}
 \dot{e}[t] &= F(t, x_d[t]) - F(t, y_r[t]) - U \\
 \text{where } U &= u[t].
 \end{aligned}
 \tag{2.8}$$

Also according to (2.5) and (2.6) equation (2.7) can be expressed as-

$$\begin{aligned}
 \dot{e}_1[t] &= a_{11}e_1[t] + \dots + a_{1k}e_k[t] + \dots + a_{1n}e_n[t] + f_1(t, x_d) - f_1(t, y_r) - u_1[t] \\
 \dot{e}_2[t] &= a_{21}e_1[t] + \dots + a_{2k}e_k[t] + \dots + a_{2n}e_n[t] + f_2(t, x_d) - f_2(t, y_r) - u_2[t] \\
 &\vdots \\
 \dot{e}_k[t] &= a_{k1}e_1[t] + \dots + a_{kk}e_k[t] + \dots + a_{kn}e_n[t] + f_k(t, x_d) - f_k(t, y_r) - u_k[t] \\
 &\vdots \\
 \dot{e}_n[t] &= a_{n1}e_1[t] + \dots + a_{nk}e_k[t] + \dots + a_{nn}e_n[t] + f_n(t, x_d) - f_n(t, y_r) - u_n[t]
 \end{aligned}
 \tag{2.9}$$

On the basis of study of chaos synchronization and control we claim that chaos exists due to nonlinear terms presents in the structure of a nonlinear dynamical system. We add control function u[t] only for nonlinear state of the system. Based on this fact for a generalized n- dimensional nonlinear dynamical system with k nonlinear functions:

$f_1(t, x_d), f_2(t, x_d), f_3(t, x_d), \dots, f_k(t, x_d); 0 \leq k \leq n$ , we decide to add control functions  $u_1[t], u_2[t], \dots, u_k[t]$  only to the respective states of non linearity.

Now (2.1) and (2.5) can be restated as identified drive system:

$$\dot{x}_d[t] = Ax_d[t] + \sum_{i=1}^k \psi_i(x_d)\varphi_i(\chi) + \theta[t]
 \tag{2.10}$$

And (2.2) and (2.8) can also be restated as identified response system:

$$\dot{y}_r[t] = Ay_r[t] + \sum_{i=1}^k \psi_i(y_r)\varphi_i(\chi) + \theta[t] + U[t]
 \tag{2.11}$$

Using (2.8) and (2.9) error system can be reformulated as-

$$\dot{e}[t] = Ae[t] + \sum_{i=1}^k \psi_i(e)\varphi_i(\chi) + U[t]
 \tag{2.12}$$

**Definition 2.1.** For arbitrary given initial points,  $(x_{1d}[t], x_{2d}[t], \dots, x_{kd}[t], \dots, x_{nd}[t])$  and  $(y_{1r}[t], y_{2r}[t], \dots, y_{kr}[t], \dots, y_{nr}[t]) \in R^n, t \in [0, \infty[$ , of the drive system (2.10) and the response system (2.11), respectively, if the solution of the error system

(2.12) has the estimation  $\sum_{i=1}^n e_i^2(t) \leq m(e[t_0]) \exp(-\alpha[t-t_0])$ , where  $m(e[t_0]) > 0$ ,

is a constant depending on the initial value  $e[t_0]$ , while  $\alpha > 0$  is a constant independent of  $e[t_0]$ , then the zero solution of the error system (2.12) is said to be globally, exponentially stable, and thus the drive-response system (2.10) and (2.11) are globally exponentially synchronized.

**Lemma 2.1.** The zero solution of the error dynamical system (2.12) is globally, exponentially stable, i.e. the drive-response systems (2.10) and (2.11) are globally exponentially synchronized, if there exists a positive definite quadratic polynomial  $V(e, t) = (e_1[t]e_2[t] \dots e_k[t] \dots e_n[t])P(e_1[t]e_2[t] \dots e_k[t] \dots e_n[t])^T$  Such that  $\dot{V}(e, t) = -(e_1[t]e_2[t] \dots e_k[t] \dots e_n[t])Q(e_1[t]e_2[t] \dots e_k[t] \dots e_n[t])^T$ . Moreover, the following negative Lyapunov exponent estimation for the error dynamical system (2.12) holds:

$$\sum_{i=1}^n e_i^2(t) \leq \frac{\lambda_{\max}(P)}{\lambda_{\min}(P)} \sum_{i=1}^n e_i^2(t_0) \exp\left[-\frac{\lambda_{\min}(Q)}{\lambda_{\max}(P)}(t-t_0)\right].$$

Where  $P = P^T \in R^{n \times n}$  and  $Q = Q^T \in R^{n \times n}$  are both positive definite matrices,  $\lambda_{\max}(P)$  and  $\lambda_{\min}(Q)$  stand for the maximum and minimum Eigen values of the matrix P and Q respectively.

Before presenting theorems, in the following we first list different types of decompositions of the nonlinear terms in (2.9) and (2.12):

$$\begin{aligned} x_{id}[t]x_{jd}[t] - y_{ir}[t]y_{jr}[t] &= \begin{cases} x_{id}[t]e_j[t] + y_{jr}[t]e_i[t] \\ y_{ir}[t]e_j[t] + x_{jd}[t]e_i[t] \end{cases} \\ &\leq \begin{cases} M_{x_{id}} |e_j[t]| + M_{y_{jr}} |e_i[t]| \\ M_{y_{ir}} |e_j[t]| + M_{x_{jd}} |e_i[t]| \end{cases} \end{aligned} \tag{2.13}$$

By using these different types of decompositions (2.13), we can obtain a number of linear feedback control laws.

### 3. Feedback control laws

**Theorem 3.1** For the given drive system (2.10) consists of k ( $0 \leq k \leq n$ ) non linear terms and the response system (2.11) with unknown controllers  $u_1[t], u_2[t], \dots, u_k[t], \dots, u_n[t]$  together with the corresponding error dynamical system (2.12). Suppose  $M_{x_{id}}$  and  $M_{y_{ir}}$  ( $i = 1, 2, 3, \dots, k, \dots, n$ ) are the upper bounds of the state variables  $x_{id}$  and  $y_{ir}$  respectively. If further linear control laws are chosen for the response system (2.11):  $u_i[t] = h_i e_i[t] \forall i = 1, 2, 3, \dots, k$  and  $u_i[t] = 0 \forall i = k + 1, k + 2, k + 3, \dots, n$ , where  $h_1, h_2, h_3, \dots, h_k$  to be determined such that derivative of Lyapunov polynomial  $V(e, t)$  is positive definite, then zero solution of the error dynamical system (2.12) is globally, exponentially stable, and thus the

drive system (2.10) and response system (2.11) are globally exponentially synchronized.

**Proof:** suppose (2.10) consists of  $k$  nonlinear terms, then response system (2.11) can be restated after using the control function  $u[t] = (u_1[t], u_2[t], u_3[t], \dots, u_k[t], \dots, u_n[t])^T$  taken as  $u_i[t] = h_i e_i[t]$   $\forall i = 1, 2, 3, \dots, k$  and  $u_i[t] = 0 \forall i = k + 1, k + 2, k + 3, \dots, n$ . Therefore

$$\dot{y}_r[t] = Ay_r[t] + \sum_{i=1}^k \psi_i(y_r) \varphi_i(\chi) + Be[t] + \theta[t] \quad (3.1.1)$$

where  $A = [a_{ij}]_{n \times n}$ ,  $B = [h_{ij}]_{k \times k}$  ( $i \neq j$ ).

Now error system (2.12) becomes

$$\dot{e}[t] = Ae[t] + \sum_{i=1}^k \psi_i(e[t]) \varphi_i(\chi) + Be[t] \quad (3.1.2)$$

If we choose Lyapunov function  $V(e, t)$  such that

$$V(e, t) = \frac{1}{2} (e_1^2[t] + e_2^2[t] + e_3^2[t] + \dots + e_k^2[t] + \dots + e_n^2[t]) \quad (3.1.3)$$

$$\text{or } V(e, t) = (e_1[t]e_2[t] \dots e_k[t] \dots e_n[t])P(e_1[t]e_2[t] \dots e_k[t] \dots e_n[t])^T \quad (3.1.4)$$

where  $P = \text{diag}[0.5, 0.5, 0.5, \dots, 0.5, \dots, 0.5]$  and  $\lambda_{\max} = \lambda_{\min} = 0.5$

Differentiating (3.1.3) with respect to  $t$  and using decompositions (2.13) we obtain

$$\dot{V}(e, t) = e_1[t]\dot{e}_1[t] + e_2[t]\dot{e}_2[t] + \dots + e_k[t]\dot{e}_k[t] + \dots + e_n[t]\dot{e}_n[t]$$

$$\dot{V}(e, t) = \begin{cases} e_1[t](a_{11}e_1[t] + a_{12}e_2[t] + \dots + a_{1k}e_k[t] + a_{1k+1}e_{k+1}[t] + \dots + a_{1n}e_n[t] \\ + f_1(t, x_d) - f_1(t, y_r) - h_1e_1[t]) \\ + e_2[t](a_{21}e_1[t] + a_{22}e_2[t] + \dots + a_{2k}e_k[t] + a_{2k+1}e_{k+1}[t] + \dots + a_{2n}e_n[t] + \\ f_2(t, x_d) - f_2(t, y_r) - h_2e_2[t]) \\ + \dots \\ + e_k[t](a_{k1}e_1[t] + a_{k2}e_2[t] + \dots + a_{kk}e_k[t] + a_{kk+1}e_{k+1}[t] + \dots + a_{kn}e_n[t] \\ + f_k(t, x_d) - f_k(t, y_r) - h_ke_k[t]) \\ + e_{k+1}[t](a_{k+11}e_1[t] + a_{k+12}e_2[t] + \dots + a_{k+1k}e_k[t] + a_{k+1k+1}e_{k+1}[t] \\ + \dots + a_{k+1n}e_n[t]) \\ + \dots \\ + e_n[t](a_{n1}e_1[t] + a_{n2}e_2[t] + \dots + a_{nk}e_k[t] + a_{nk+1}e_{k+1}[t] + \dots + a_{nn}e_n[t]) \end{cases}$$

$$\begin{aligned}
 & \left( \begin{aligned}
 & (a_{11} - h_1)e_1^2[t] + a_{12}e_1[t]e_2[t] + \dots + a_{1k}e_1[t]e_k[t] \\
 & + a_{1k+1}e_1[t]e_{k+1}[t] + \dots + a_{1n}e_1[t]e_n[t] + e_1[t](f_1(t, x_d) - f_1(t, y_r)) \\
 & + a_{21}e_2[t]e_1[t] + (a_{22} - h_2)e_2^2[t] + \dots + a_{2k}e_2[t]e_k[t] \\
 & + a_{2k+1}e_2[t]e_{k+1}[t] + \dots + a_{2n}e_2[t]e_n[t] + e_2[t](f_2(t, x_d) - f_2(t, y_r)) \\
 & + \dots \\
 & + a_{k1}e_k[t]e_1[t] + a_{k2}e_k[t]e_2[t] + \dots + (a_{kk} - h_k)e_k^2[t] \\
 & + a_{kk+1}e_k[t]e_{k+1}[t] + \dots + a_{kn}e_k[t]e_n[t] + e_k[t](f_k(t, x_d) - f_k(t, y_r)) \\
 & + a_{k+11}e_{k+1}[t]e_1[t] + a_{k+12}e_{k+1}[t]e_2[t] + \dots + a_{k+1k}e_{k+1}[t]e_k[t] \\
 & + a_{k+1k+1}e_{k+1}^2[t] \\
 & + \dots \\
 & + a_{n1}e_n[t]e_1[t] + a_{n2}e_n[t]e_2[t] + \dots + a_{nk}e_n[t]e_k[t] + a_{nk+1}e_n[t]e_{k+1}[t]
 \end{aligned} \right. \tag{3.1.5}
 \end{aligned}$$

Now, if we consider the nonlinear functions  $f_i(t, x_d) - f_i(t, y_r) = g_i(e[t], x_d, y_r)$  as follows

$$f_i(t, x_d) - f_i(t, y_r) \leq g_i(e[t], M_{x_d}, M_{y_r}); (i = 1, 2, 3, \dots, k) \tag{3.1.6}$$

Using (3.1.6) we obtain the following inequality

$$\begin{aligned}
 \dot{V}(e, t) = & \left( \begin{aligned}
 & (a_{11} - h_1)e_1^2[t] + a_{12}e_1[t]e_2[t] + \dots + a_{1k}e_1[t]e_k[t] \\
 & + a_{1k+1}e_1[t]e_{k+1}[t] + \dots + a_{1n}e_1[t]e_n[t] + e_1[t](g_1(e[t], x_d, y_r)) \\
 & + a_{21}e_2[t]e_1[t] + (a_{22} - h_2)e_2^2[t] + \dots + a_{2k}e_2[t]e_k[t] \\
 & + a_{2k+1}e_2[t]e_{k+1}[t] + \dots + a_{2n}e_2[t]e_n[t] + e_2[t](g_2(e[t], x_d, y_r)) \\
 & + \dots \\
 & + a_{k1}e_k[t]e_1[t] + a_{k2}e_k[t]e_2[t] + \dots + (a_{kk} - h_k)e_k^2[t] \\
 & + a_{kk+1}e_k[t]e_{k+1}[t] + \dots + a_{kn}e_k[t]e_n[t] + e_k[t](g_k(e[t], x_d, y_r)) \\
 & + a_{k+11}e_{k+1}[t]e_1[t] + a_{k+12}e_{k+1}[t]e_2[t] \\
 & + \dots + a_{k+1k}e_{k+1}[t]e_k[t] + a_{k+1k+1}e_{k+1}^2[t] \\
 & + \dots \\
 & + a_{n1}e_n[t]e_1[t] + a_{n2}e_n[t]e_2[t] \\
 & + \dots + a_{nk}e_n[t]e_k[t] + a_{nk+1}e_n[t]e_{k+1}[t]
 \end{aligned} \right.
 \end{aligned}$$

$$\leq \left\{ \begin{array}{l} (a_{11} - h_1)e_1^2[t] + a_{12}e_1[t]e_2[t] + \dots + a_{1k}e_1[t]e_k[t] \\ + a_{1k+1}e_1[t]e_{k+1}[t] + \dots + a_{1n}e_1[t]e_n[t] + e_1[t](g_1(e[t], M_{x_d}, M_{y_r})) \\ + a_{21}e_2[t]e_1[t] + (a_{22} - h_2)e_2^2[t] + \dots + a_{2k}e_2[t]e_k[t] \\ + a_{2k+1}e_2[t]e_{k+1}[t] + \dots + a_{2n}e_2[t]e_n[t] + e_2[t](g_2(e[t], M_{x_d}, M_{y_r})) \\ + \dots \\ + a_{k1}e_k[t]e_1[t] + a_{k2}e_k[t]e_2[t] + \dots + (a_{kk} - h_k)e_k^2[t] \\ + a_{kk+1}e_k[t]e_{k+1}[t] + \dots + a_{kn}e_k[t]e_n[t] + e_k[t](g_k(e[t], M_{x_d}, M_{y_r})) \\ + a_{k+11}e_{k+1}[t]e_1[t] + a_{k+12}e_{k+1}[t]e_2[t] + \\ + \dots + a_{k+1k}e_{k+1}[t]e_k[t] + a_{k+1k+1}e_{k+1}^2[t] \\ + \dots \\ + a_{n1}e_n[t]e_1[t] + a_{n2}e_n[t]e_2[t] \\ + \dots + a_{nk}e_n[t]e_k[t] + a_{nk+1}e_n[t]e_{k+1}[t] \end{array} \right.$$

Since the term corresponding to each  $g_1, g_2, g_3, \dots, g_k$  can be arranged in such a way that polynomial  $\dot{V}(e, t)$  is positive definite

$$\dot{V}(e, t) = (e_1[t]e_2[t] \dots e_k[t] \dots e_n[t])Q(e_1[t]e_2[t] \dots e_k[t] \dots e_n[t])^T \tag{3.1.7}$$

where Q is positive definite matrix given by-

$$Q = \begin{bmatrix} a_{11} - h_1 & \frac{1}{2}(a_{12} + M_{x_{12}} + M_{y_{12}}) & \dots & \frac{1}{2}(a_{1k} + M_{x_{1k}} + M_{y_{1k}}) & \dots & \frac{1}{2}(a_{1n} + M_{x_{1n}} + M_{y_{1n}}) \\ \frac{1}{2}(a_{21} + M_{x_{21}} + M_{y_{21}}) & a_{22} - h_2 & \dots & \frac{1}{2}(a_{2k} + M_{x_{2k}} + M_{y_{2k}}) & \dots & \frac{1}{2}(a_{2n} + M_{x_{2n}} + M_{y_{2n}}) \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \frac{1}{2}(a_{k1} + M_{x_{k1}} + M_{y_{k1}}) & \frac{1}{2}(a_{k2} + M_{x_{k2}} + M_{y_{k2}}) & \dots & a_{kk} - h_k & \dots & \frac{1}{2}(a_{kn} + M_{x_{kn}} + M_{y_{kn}}) \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \frac{1}{2}(a_{n1} + M_{x_{n1}} + M_{y_{n1}}) & \frac{1}{2}(a_{n2} + M_{x_{n2}} + M_{y_{n2}}) & \dots & \frac{1}{2}(a_{nk} + M_{x_{nk}} + M_{y_{nk}}) & \dots & a_{nn} \end{bmatrix}$$

Hence by lemma (2.1) zero solution of the error system (3.1.2) is globally; exponentially stable. Proof of the Theorem (3.1) is complete.

### 4. Controlling Lorenz system

The Lorenz system described by the following system of non linear differential equations

$$\begin{aligned}\dot{x}[t] &= \sigma(y[t] - x[t]) \\ \dot{y}[t] &= rx[t] - y[t] - x[t]z[t] \\ \dot{z}[t] &= -bz[t] + x[t]y[t]\end{aligned}$$

for the parameter values  $\sigma = 10, r = 28, b = \frac{8}{3}$  has a chaotic attractor portrayed in fig. (4.1)

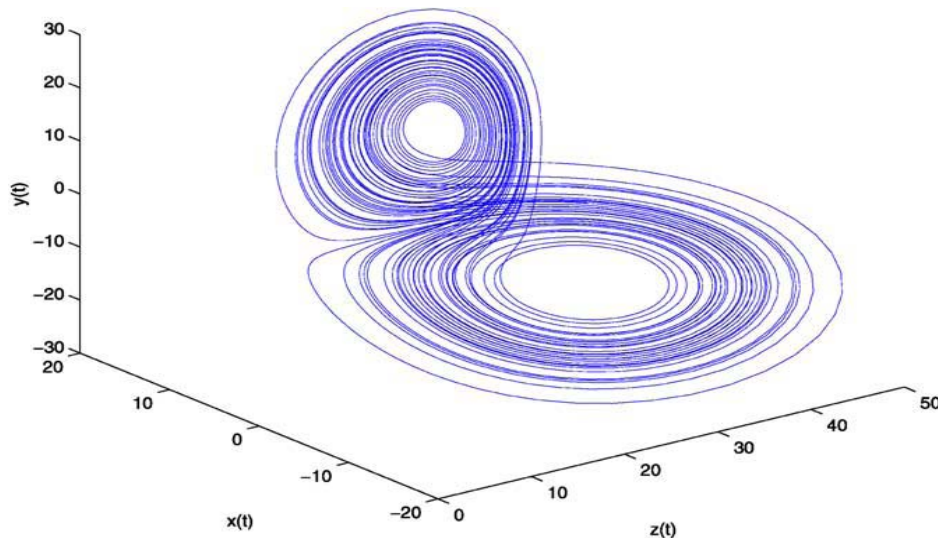


Fig-4.1. Chaotic attractor of Lorenz system

Now, consider the Lorenz chaotic system

$$\begin{aligned}\dot{x}_{1d}[t] &= \sigma(x_{2d}[t] - x_{1d}[t]) \\ \dot{x}_{2d}[t] &= rx_{1d}[t] - x_{2d}[t] - x_{1d}[t]x_{3d}[t] \\ \dot{x}_{3d}[t] &= -bx_{3d}[t] + x_{1d}[t]x_{2d}[t]\end{aligned} \tag{4.1}$$

As a drive system and the response system given by

$$\begin{aligned}\dot{y}_{1r}[t] &= \sigma(y_{2r}[t] - y_{1r}[t]) + u_1[t] \\ \dot{y}_{2r}[t] &= ry_{1r}[t] - y_{2r}[t] - y_{1r}[t]y_{3r}[t] + u_2[t] \\ \dot{y}_{3r}[t] &= -by_{3r}[t] + y_{1r}[t]y_{2r}[t] + u_3[t]\end{aligned} \tag{4.2}$$

where  $u_i[t]$  ( $i = 1, 2, 3$ ) are unknown feedback controlling functions of t. If

$$e[t] = (e_1[t], e_2[t], e_3[t])^T = (x_{1d}[t] - y_{1r}[t], x_{2d}[t] - y_{2r}[t], x_{3d}[t] - y_{3r}[t])^T.$$

Then from (4.1) and (4.2), we obtain the following four error dynamical systems:

$$\begin{aligned}\dot{e}_1[t] &= \sigma(e_2[t] - e_1[t]) - u_1[t] \\ \dot{e}_2[t] &= re_1[t] - e_2[t] - x_{1d}[t]e_3[t] - y_{3r}e_1[t] - u_2[t]\end{aligned} \tag{4.3a}$$

$$\begin{aligned}\dot{e}_3[t] &= -be_3[t] + x_{1d}[t]e_2[t] + y_{2r}e_1[t] - u_3[t] \\ \dot{e}_1[t] &= \sigma(e_2[t] - e_1[t]) - u_1[t] \\ \dot{e}_2[t] &= re_1[t] - e_2[t] - x_{1d}[t]e_3[t] - y_{3r}e_1[t] - u_2[t] \\ \dot{e}_3[t] &= -be_3[t] + x_{2d}[t]e_1[t] + y_{1r}e_2[t] - u_3[t]\end{aligned} \tag{4.3b}$$

$$\begin{aligned}\dot{e}_1[t] &= \sigma(e_2[t] - e_1[t]) - u_1[t] \\ \dot{e}_2[t] &= re_1[t] - e_2[t] - x_{3d}[t]e_1[t] - y_{1r}e_3[t] - u_2[t]\end{aligned}\quad (4.3c)$$

$$\begin{aligned}\dot{e}_3[t] &= -be_3[t] + x_{1d}[t]e_2[t] + y_{2r}e_1[t] - u_3[t] \\ \dot{e}_1[t] &= \sigma(e_2[t] - e_1[t]) - u_1[t] \\ \dot{e}_2[t] &= re_1[t] - e_2[t] - x_{3d}[t]e_1[t] - y_{1r}e_3[t] - u_2[t]\end{aligned}\quad (4.3d)$$

$$\dot{e}_3[t] = -be_3[t] + x_{2d}[t]e_1[t] + y_{1r}e_2[t] - u_3[t]$$

Now theorem (3.1) for a particular case (Lorenz system) can be restated as follow-  
**Theorem 4.1** For the drive system (4.1) consists of two non linear terms and the response system (4.2) with unknown controllers  $u_1[t], u_2[t], u_3[t]$  together with the corresponding error dynamical systems (4.3a)-(4.3d). Suppose  $M_{x_{id}}$  and

$M_{y_{ir}}$  ( $i=1,2,3$ ) are the upper bounds of the state variables  $x_{id}$  and  $y_{ir}$  respectively.

If further linear control laws are chosen for the response system (4.2):  $u_1[t] = 0, u_2[t] = h_2e_2[t], u_3[t] = h_3e_3[t]$ , where  $h_2, h_3$  to be determined such that derivative of Lyapunov polynomial  $V(e, t)$  is positive definite, then zero solution of the error dynamical system (4.3a)-(4.3d) is globally, exponentially stable, and thus the drive system (4.1) and response system (4.2) are globally exponentially synchronized.

**Proof:** Since the drive system (4.1) consists of two nonlinear terms, therefore response system (4.2) can be restated after taking control functions  $u[t] = (u_1[t], u_2[t], u_3[t])^T$  as  $u_1[t] = 0, u_2[t] = h_2e_2[t], u_3[t] = h_3e_3[t]$ , as

$$\dot{y}_r[t] = Ay_r[t] + \sum_{i=1}^2 \psi_i(y_r)\phi_i(\chi) + Be[t] + \theta[t] \quad (4.1.1)$$

where

$$A = \begin{bmatrix} -\sigma & \sigma & 0 \\ r & -1 & 0 \\ 0 & 0 & b \end{bmatrix}, \quad \phi(\chi) = x_1, \quad \psi(y_r) = [0, -y_{3r}, y_{2r}]^T, \quad \theta(t) = [0, 0, 0]^T,$$

$$B = [0, h_2, h_3]^T.$$

Now error system (4.3a)-(4.3d) become

$$\dot{e}[t] = Ae[t] + \sum_{i=1}^2 \psi(e[t])\phi_i(\chi) + Be[t] \quad (4.1.2)$$

if we choose Lyapunov function

$$V(e, t) = \frac{1}{2}(e_1^2[t] + e_2^2[t] + e_3^2[t])^T \quad (4.1.3)$$

or

$$V(e, t) = (e_1[t]e_2[t]e_3[t])P(e_1[t]e_2[t]e_3[t])^T \quad (4.1.4)$$

where  $P = \text{diag}[0.5, 0.5, 0.5]$  and

$$\lambda_{\max} = \lambda_{\min} = 0.5.$$

Differentiating (4.1.3) with respect to t and using decompositions we obtain

$$\begin{aligned} \dot{V}(e, t) &= e_1[t]\dot{e}_1[t] + e_2[t]\dot{e}_2[t] + e_3[t]\dot{e}_3[t] \\ &= \begin{cases} -\sigma e_1^2[t] - (1+h_2)e_2^2[t] - (b+h_3)e_3^2[t] + (\sigma+r-y_3)e_1[t]e_2[t] + y_2e_1[t]e_3[t] \\ -\sigma e_1^2[t] - (1+h_2)e_2^2[t] - (b+h_3)e_3^2[t] + (\sigma+r-y_3)e_1[t]e_2[t] + x_2e_1[t]e_3[t] \\ +(-x_1+y_1)e_2[t]e_3[t] \\ -\sigma e_1^2[t] - (1+h_2)e_2^2[t] - (b+h_3)e_3^2[t] + (\sigma+r-x_3)e_1[t]e_2[t] + y_2e_1[t]e_3[t] \\ +(-y_1+x_1)e_2[t]e_3[t] \\ -\sigma e_1^2[t] - (1+h_2)e_2^2[t] - (b+h_3)e_3^2[t] + (\sigma+r-x_3)e_1[t]e_2[t] + x_2e_1[t]e_3[t] \\ -(\sigma e_1^2[t] + (1+h_2)e_2^2[t] + (b+h_3)e_3^2[t] - (\sigma+r+M_{y_3})|e_1[t]||e_2[t]| \\ -M_{y_2}|e_1[t]||e_3[t]|), \\ -(\sigma e_1^2[t] + (1+h_2)e_2^2[t] + (b+h_3)e_3^2[t] - (\sigma+r+M_{y_3})|e_1[t]||e_2[t]| \\ -M_{x_2}|e_1[t]||e_3[t]| - (M_{x_1}+M_{y_1})|e_2[t]||e_3[t]|), \\ -(\sigma e_1^2[t] + (1+h_2)e_2^2[t] + (b+h_3)e_3^2[t] - (\sigma+r+M_{x_3})|e_1[t]||e_2[t]| \\ -M_{y_2}e_1[t]e_3[t] - (M_{x_1}+M_{y_1})|e_2[t]||e_3[t]|), \\ -(\sigma e_1^2[t] + (1+h_2)e_2^2[t] + (b+h_3)e_3^2[t] - (\sigma+r+M_{x_3})|e_1[t]||e_2[t]| \\ -M_{x_2}|e_1[t]||e_3[t]|) \end{cases} \\ &\leq \begin{cases} -(\sigma e_1^2[t] + (1+h_2)e_2^2[t] + (b+h_3)e_3^2[t] - (\sigma+r+M_{y_3})|e_1[t]||e_2[t]| \\ -M_{y_2}|e_1[t]||e_3[t]|), \\ -(\sigma e_1^2[t] + (1+h_2)e_2^2[t] + (b+h_3)e_3^2[t] - (\sigma+r+M_{y_3})|e_1[t]||e_2[t]| \\ -M_{x_2}|e_1[t]||e_3[t]| - (M_{x_1}+M_{y_1})|e_2[t]||e_3[t]|), \\ -(\sigma e_1^2[t] + (1+h_2)e_2^2[t] + (b+h_3)e_3^2[t] - (\sigma+r+M_{x_3})|e_1[t]||e_2[t]| \\ -M_{y_2}e_1[t]e_3[t] - (M_{x_1}+M_{y_1})|e_2[t]||e_3[t]|), \\ -(\sigma e_1^2[t] + (1+h_2)e_2^2[t] + (b+h_3)e_3^2[t] - (\sigma+r+M_{x_3})|e_1[t]||e_2[t]| \\ -M_{x_2}|e_1[t]||e_3[t]|) \end{cases} \\ &= \begin{cases} -(|e_1[t]||e_2[t]||e_3[t]|)Q_1(|e_1[t]||e_2[t]||e_3[t]|)^T, \\ -(|e_1[t]||e_2[t]||e_3[t]|)Q_2(|e_1[t]||e_2[t]||e_3[t]|)^T, \\ -(|e_1[t]||e_2[t]||e_3[t]|)Q_3(|e_1[t]||e_2[t]||e_3[t]|)^T, \\ -(|e_1[t]||e_2[t]||e_3[t]|)Q_4(|e_1[t]||e_2[t]||e_3[t]|)^T \end{cases} \end{aligned}$$

where

$$Q_1 = \begin{bmatrix} \sigma & -\frac{1}{2}(\sigma+r+M_{y_3}) & -\frac{1}{2}M_{y_2} \\ -\frac{1}{2}(\sigma+r+M_{y_3}) & 1+h_2 & 0 \\ -\frac{1}{2}M_{y_2} & 0 & b+h_3 \end{bmatrix}$$

$$Q_2 = \begin{bmatrix} \sigma & -\frac{1}{2}(\sigma+r+M_{y_3}) & -\frac{1}{2}M_{x_2} \\ -\frac{1}{2}(\sigma+r+M_{y_3}) & 1+h_2 & -\frac{1}{2}(M_{x_1}+M_{y_1}) \\ -\frac{1}{2}M_{x_2} & -\frac{1}{2}(M_{x_1}+M_{y_1}) & b+h_3 \end{bmatrix}$$

$$Q_3 = \begin{bmatrix} \sigma & -\frac{1}{2}(\sigma + r + M_{x_3}) & -\frac{1}{2}M_{y_2} \\ -\frac{1}{2}(\sigma + r + M_{x_3}) & 1 + h_2 & -\frac{1}{2}(M_{x_1} + M_{y_1}) \\ -\frac{1}{2}M_{y_2} & -\frac{1}{2}(M_{x_1} + M_{y_1}) & b + h_3 \end{bmatrix}$$

$$Q_4 = \begin{bmatrix} \sigma & -\frac{1}{2}(\sigma + r + M_{x_3}) & -\frac{1}{2}M_{x_2} \\ -\frac{1}{2}(\sigma + r + M_{x_3}) & 1 + h_2 & 0 \\ -\frac{1}{2}M_{x_2} & 0 & b + h_3 \end{bmatrix}$$

It is easy to see that the zero solution of these error systems is globally exponentially stable if the symmetric matrices  $Q_i$  ( $i = 1, 2, 3, 4$ ) are positive definite, for which the following conditions must hold:

$$Q_1 = \begin{cases} \sigma > 0 \\ h_2 > \frac{1}{4\sigma}(\sigma + r + M_{y_3})^2 - 1 \\ h_3 > \frac{M_{y_2}^2}{4\sigma - \left[\frac{(\sigma + r + M_{y_3})^2}{1 + h_2}\right]} - b \end{cases}$$

$$Q_2 = \begin{cases} \sigma > 0 \\ h_2 > \frac{1}{4\sigma}(\sigma + r + M_{y_3})^2 - 1 \\ h_3 > \frac{1}{4} \left[ \frac{\sigma(M_{x_1} + M_{y_1})^2 + M_{x_2}^2(1 + h_2) + (M_{x_1}M_{x_2} + M_{x_2}M_{y_1})(\sigma + r + M_{y_3})}{\sigma(1 + h_2) - \frac{1}{4}(\sigma + r + M_{y_3})^2} \right] - b \end{cases}$$

$$Q_3 = \begin{cases} \sigma > 0 \\ h_2 > \frac{1}{4\sigma}(\sigma + r + M_{x_3})^2 - 1 \\ h_3 > \left[ \frac{\sigma(M_{x_1} + M_{y_1})^2 + M_{y_2}^2(1 + h_2) + (M_{x_1}M_{y_2} + M_{y_1}M_{x_2})(\sigma + r + M_{x_3})}{4\sigma(1 + h_2) - (\sigma + r + M_{x_3})^2} \right] - b \end{cases}$$

$$Q_4 = \begin{cases} \sigma > 0 \\ h_2 > \frac{1}{4\sigma}(\sigma + r + M_{x_3})^2 - 1 \\ h_3 > \left[ \frac{M_{x_2}^2(1 + h_2)}{4\sigma(1 + h_2) - (\sigma + r + M_{x_3})^2} - b \right] \end{cases}$$

Now by using lemma (2.1), we have the exponential estimation for

$Q_i (i = 1, 2, 3, 4)$ .

$$\sum_{i=1}^3 e_i^2(t) \leq \frac{\lambda_{\max}(P)}{\lambda_{\min}(P)} \sum_{i=1}^3 e_i^2(t_0) \exp\left[-\frac{\lambda_{\min}(Q_i)}{\lambda_{\max}(P)}(t - t_0)\right].$$

Proof of theorem (4.1) is complete.

### 5. Numerical simulation results

In this section, we verify the control laws presented in the previous sections via numerical simulations. We take the parameter values as  $\sigma = 16$ ,  $b = 4$ ,  $r = 45.6$  and initial conditions-

$$x_{d_{10}} = 2, x_{d_{20}} = 3.5, x_{d_{30}} = 18.4 \text{ and } y_{r_{10}} = -1, y_{r_{20}} = 10.2, y_{r_{30}} = 8.3.$$

Error dynamics for the synchronized Lorenz system between  $x_{d_i}$  and  $y_{r_i}$  ( $i = 1, 2, 3$ ) are in the fig. (5.1).

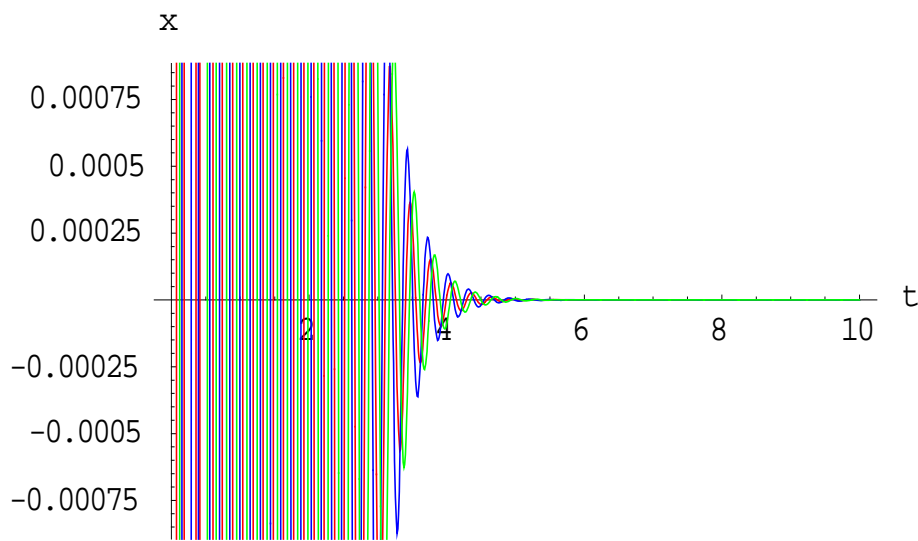


Fig 5.1  $x = x[t]$ ,  $y = y[t]$ ,  $z = z[t]$  for  $0 \leq t \leq 10$

## 6. Conclusions

In this paper, we focus to study the chaos synchronization of generalized nonlinear dynamical systems which can exhibit a Chaotic attractor for different parameter values by using Linear Feedback control Laws. We propose a theorem for synchronization of an n-dimensional generalized nonlinear dynamical system. Numerical simulations are used to verify the theoretical results.

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