On an Elliptic Boundary Problem Governed by a Laplacian Operator Perturbed by a Spectral Parameter and Affected by a Weight Function in a Plane Polygon

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Abstract
In this work, we study the problem governed by the laplacian operator perturbed by a spectral parameter and affected by a weight function in a polygon. First, the study will be done in a plane sector. We will show a theorem that links the study of this problem and a problem of Poisson. This study is it provides original results in a richer functional space, more it covers the classical case.

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1 Notations. $\Omega$ denoted a bounded open of $\mathbb{R}^2$ with boundary $\Gamma$. Here $\Gamma$ is a polygon, $\phi_i; 1 \leq i \leq N$ denoted the angles of this polygon, and $\Omega_\phi$ denoted a plane sector, it will be defined by the following expression: $\Omega_\phi = \{(r, \theta) / r > 0; \ 0 < \theta < \phi < 2\pi\}$ its boundary is $\Gamma_\phi = \Gamma_1 \cup \Gamma_2$ where $\Gamma_1$ and $\Gamma_2$ are two half lines with the respective equations: $\theta = 0; \ \theta = \phi$
2 Introduction

One shows in [1] ; [3] that if $\Omega$ is a regular open domain, the following problem:

$$\begin{cases}
\Delta u + \lambda u = f & \text{in } \Omega \\
u = 0 & \text{on } \Gamma
\end{cases}$$

(2.1)

admits an unique solution in $H^2(\Omega)$, where $\lambda$ is a real parameter (called spectral parameter) and $f$ given, this result is false if $\Omega$ is not a regular domain, for example if the boundary of $\Omega$ is a polygon, because the boundary of $\Omega$ admits angular points. To recover this, we multiply the operator $\Delta + \lambda$ by a weight function $\mu$ which is continuous and not null in $\Omega$, and null in the neighborhood of each top of $\Omega$, and it does not change the physical aspect of the problem.

In the first step the study of this problem will be made in a plane sector $\Omega_\varphi$, by rotation and by translation we can always bring its top to the origin in local coordinates. The polar coordinates are adapted better to the geometry of $\Omega_\varphi$.

**Remark 2.1:** - This weight function absorbs the singularities which occur in the neighborhood of the top of $\Omega_\varphi$, also it makes calculations easier.

Considering the presence of the angular point on this top, to recover this, we multiply the operator $\Delta + \lambda$ by the function $\mu(r)$ which satisfies $\mu(r) = r$ in the neighborhood of the origin.

3 Position of the Problem

We propose to study the following problem:

$$\begin{cases}
\mu(r)(\Delta u + \lambda u) = \mu(r)f & \text{in } \Omega_\varphi \\
u = 0 & \text{on } \Gamma_\varphi
\end{cases}$$

(3.1)

In the neighborhood of the origin, the problem (3.1) becomes:

$$\begin{cases}
Lu = r(\Delta u + \lambda u) = rf & \text{in } \Omega_\varphi \\
u = 0 & \text{on } \Gamma_\varphi
\end{cases}$$

(3.2)

We pose $x = r \cos \theta; \ y = r \sin \theta$, therefore the problem (3.2) becomes:

$$\begin{cases}
r \frac{\partial^2 u}{\partial r^2} + \frac{\partial u}{\partial r} + \frac{1}{r} \frac{\partial^2 u}{\partial \theta^2} + \lambda ru = rf & \text{in } \Omega_\varphi \\
u(r,0) = 0
\end{cases}$$

(3.3)

$$u(r \cos \varphi, r \sin \varphi) = 0$$
The choice of the weight function $\mu(r)$ gives us the idea to seek the solutions of the problem (3.3) in a new functional space which we denote $E^2(\Omega_\varphi)$ and we defined it by the following expression:

$$E^2(\Omega_\varphi) = \{ u \in H^1(\Omega_\varphi) \text{ such that } rD^\alpha u \in L^2(\Omega_\varphi) \}$$

where the support of the function $u$ is compact in $\overline{\Omega_\varphi}$, and $D$ denote the operator of derivation compared to the variables $r$ and $\eta$, $\eta$ being the normal vector to $\Gamma_\varphi$ directed towards the outside of $\Omega_\varphi$, and $\frac{\partial}{\partial \eta} = \frac{1}{r} \frac{\partial}{\partial \theta}$.

We pose $E^2_0(\Omega_\varphi) = E^2(\Omega_\varphi) \cap H^1_0(\Omega_\varphi)$.

We definite the norm of $E^2(\Omega_\varphi)$ as

$$\|u\|_{E^2(\Omega_\varphi)}^2 = \|u\|^2_{H^1(\Omega_\varphi)} + \sum_{|\alpha| \leq 2} \|rD^\alpha u\|^2_{L^2(\Omega_\varphi)}$$

3.1 Property of Space $E^2(\Omega_\varphi)$ We showed in [4], the following results:

$$H^2_{x,y}(\Omega_\varphi) \subset E^2(\Omega_\varphi) \subset H^1_{x,y}(\Omega_\varphi)$$

$$H^1_{x,y}(\Omega_\varphi) = H^1_{r,\eta}(\Omega_\varphi)$$

$$H^2_{x,y}(\Omega_\varphi) \subset H^2_{r,\eta}(\Omega_\varphi)$$

4 Study of the Existence of the solution

We propose to study the solution of the problem (3.2) in plane sector $\Omega_\varphi$. This study will be extended to the polygon by using a partition of the unity of this one.

**Theorem 4.1.-** If $f$ is a function given in $L^2(\Omega_\varphi)$, the problem (3.2) admits an unic solution in space $E^2(\Omega_\varphi)$ if and only if the following problem:

$$\begin{cases}
\Delta u = f & \text{in } \Omega_\varphi \\
u = 0 & \text{on } \Gamma_\varphi
\end{cases}$$

admits an unic solution in space $E^2(\Omega_\varphi)$.

**Proof.-** It is evident that if the Problem (3.2) admits an unic solution in space $E^2(\Omega_\varphi)$, then the Problem (4.1) admits an unic solution in space $E^2(\Omega_\varphi)$, because the problem (4.1) is a particular case of the problem (3.2), it suffices to pose $\lambda = 0$ in (3.2). We will prove the reciprocal.

We propose to seek the solution if it exists for the problem (4.1) in the space $E^2(\Omega_\varphi)$. To study the existence of the solution of this problem, we will study the...
image of the operator \( A = r \Delta \) in space \( L^2(\Omega_\varphi) \). To establish that the image of \( A \) is closed in \( L^2(\Omega_\varphi) \), we will be the following inequality: There is a constant \( K > 0 \) such that:

\[
\| u \|^2_{E^1(\Omega_\varphi)} \leq K \left\{ \| A u \|^2_{E^1(\Omega_\varphi)} + \| u \|^2_{H^1(\Omega_\varphi)} \right\} \tag{4.2}
\]

The inequality (4.2), is deduced from the following lemma showed in [2]

**Lemma 4.1** (of Peetre). If \( X, Y, Z \) are three Banach reflexive spaces such that \( X \subset Y \) with compact injection, and \( P \) is a continuous linear operator from \( X \) to \( Z \), then the following conditions are equivalent:

(I) the image of \( P \) is closed in \( Z \) and the kernel of \( P \) has a finite dimension.

(II) there is constant \( C > 0 \) such that:

\[
\| U \|_X \leq C \left\{ \| P U \|_Y + \| U \|_X \right\}.
\]

For our problem we consider \( P = A; X = E^2(\Omega_\varphi); Y = H^1(\Omega_\varphi) \) and \( Z = L^2(\Omega_\varphi) \).

We showed in [5], that all the hypothesis of the Lemma 4.1 are verified, so to obtain (I) in the Lemma 4.1, we will prove (II), then we will establish the inequality (4.2). For this we need the following Lemma .

**Lemme 4.2**.- For each function \( u \in L^2(\Omega_\varphi) \) we have:

\[
\left\| \frac{\partial^2 u}{\partial r^2} \right\|_{L^2(\Omega_\varphi)} \geq \left\| \frac{\partial u}{\partial r} \right\|_{L^2(\Omega_\varphi)}
\]

**Proof**

\[
0 \leq \left\| \frac{\partial^2 u}{\partial r^2} + \frac{\partial u}{\partial r} \right\|^2_{L^2(\Omega_\varphi)} = \left\| \frac{\partial^2 u}{\partial r^2} \right\|^2_{L^2(\Omega_\varphi)} + \left\| \frac{\partial u}{\partial r} \right\|^2_{L^2(\Omega_\varphi)} + 2 \int_0^r \int_0^r \frac{\partial^2 u}{\partial r^2} \frac{\partial u}{\partial r} r drd\theta
\]

We calculate \( \int_0^\infty \int_0^r r \frac{\partial^2 u}{\partial r^2} \frac{\partial u}{\partial r} r drd\theta \) using integration by parts, according to the support of the function \( u \) is compact and the boundary values, we deduce that:

\[
\left\| \frac{\partial^2 u}{\partial r^2} + \frac{\partial u}{\partial r} \right\|^2_{L^2(\Omega_\varphi)} = \left\| \frac{\partial^2 u}{\partial r^2} \right\|^2_{L^2(\Omega_\varphi)} + \left\| \frac{\partial u}{\partial r} \right\|^2_{L^2(\Omega_\varphi)} \Rightarrow \left\| \frac{\partial^2 u}{\partial r^2} \right\|^2_{L^2(\Omega_\varphi)} \geq \left\| \frac{\partial u}{\partial r} \right\|^2_{L^2(\Omega_\varphi)}
\]

We prove now the inequality (4.2).

\[
\left\| r \Delta u \right\|^2_{L^2(\Omega_\varphi)} = \left\| \frac{\partial^2 u}{\partial r^2} \right\|^2_{L^2(\Omega_\varphi)} + \left\| \frac{\partial u}{\partial r} \right\|^2_{L^2(\Omega_\varphi)} + \frac{1}{r} \left\| \frac{\partial^2 u}{\partial r \partial \theta^2} \right\|^2_{L^2(\Omega_\varphi)} + 2 I J K
\]

Where \( I = \int_0^\infty \int_0^r r \frac{\partial^2 u}{\partial r^2} \frac{\partial u}{\partial r} r drd\theta \); \( J = \int_0^\infty \int_0^r \frac{\partial^2 u}{\partial r^2} \frac{1}{r} \frac{\partial^2 u}{\partial \theta^2} r drd\theta \)
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\[ K = \int_0^r \int_0^\vartheta \frac{\partial u}{\partial r} \frac{1}{r} \frac{\partial^2 u}{\partial \vartheta^2} \, r \, dr \, d\vartheta . \]

According to the lemma 4.2, we find \( I = -\left\| \frac{\partial u}{\partial r} \right\|_{L^2(\varOmega_\psi)}^2 \). Now, we calculate \( J \) using integration by part, according to the boundary values and the support of the function \( u \) is compact, we deduce that:

\[ J = -\int_0^r \int_0^\vartheta \frac{\partial u}{\partial r} \frac{\partial^3 u}{\partial r \partial \vartheta^2} \, r \, dr \, d\vartheta - K \]

Calculating the integral \( \int_0^\vartheta \frac{\partial u}{\partial r} \frac{\partial^3 u}{\partial r \partial \vartheta^2} \, r \, dr \, d\vartheta \), using integration by parts, we obtain:

\[ J = \int_0^r \int_0^\vartheta \frac{\partial^2 u}{\partial r \partial \vartheta} \, r \, dr \, d\vartheta \]

Therefore:

\[ J = \int_0^r \int_0^\vartheta \frac{\partial^2 u}{\partial r \partial \vartheta} \, r \, dr \, d\vartheta - K \]

According to the lemma 4.2 and the definition (3.5), we deduce that:

\[ 2 \left( \frac{\partial^3 u}{\partial r \partial \vartheta} \right)_{\varOmega(\varpsi)} + \left\| u_{e(\varpsi)} \right\| \geq \left\| u_{e(\varpsi)} \right\| + \left\| u_{e(\varpsi)} \right\| - \frac{\partial u}{\partial r} \left\| u_{e(\varpsi)} \right\| + \left\| u_{e(\varpsi)} \right\| + \frac{\partial^2 u}{\partial r \partial \vartheta} \left\| u_{e(\varpsi)} \right\| + \left\| u_{e(\varpsi)} \right\| \]

We remark that \( \left\| u_{e(\varpsi)} \right\| \geq \frac{\partial u}{\partial r} \left\| u_{e(\varpsi)} \right\| \). Then:

\[ 2 \left\| u_{e(\varpsi)} \right\| \leq 2 \left\| \frac{\partial^3 u}{\partial r \partial \vartheta} \right\| \left\| u_{e(\varpsi)} \right\| \]

Thus according to inequality (4.3) and the lemma (4.1), we deduce that the problem (4.1) admit a solution in \( E^2(\varOmega_\psi) \).

Now we prove that the inequality (4.3) can be extented to the problem (3.2).

\[ \left\| Lu \right\| = \left\| -r \Delta u + r \lambda u \right\| = \left\| r \Delta u \right\| + \lambda^2 \left\| u \right\|^2 - 2\lambda \left\langle r \Delta u, ru \right\rangle \]

\[ \left\| Lu \right\| = \left\| -r \Delta u + r \lambda u \right\| = \left\| r \Delta u \right\| + \lambda^2 \left\| u \right\|^2 - 2\lambda \left\langle r \Delta u, ru \right\rangle \]

We calculate \( \left\langle r \Delta u, ru \right\rangle = \int_0^r \int_0^\vartheta r \Delta u \, r \, dr \, d\vartheta = \int_0^r \int_0^\vartheta r^3 \Delta u \, dr \, d\vartheta \)

We know that the support of the function \( u \) is compact in \( \Omega_\psi \), therefore it exist a finite number \( r_1 > 0 \), such that:

\[ \int_0^r \int_0^\vartheta r^3 \Delta u \, dr \, d\vartheta = \int_0^r \int_0^\vartheta r^3 \Delta u \, dr \, d\vartheta \leq r_1^3 \int_0^r \int_0^\vartheta \Delta u \, dr \, d\vartheta \]

It is easier to verify that:

\[ \int_0^r \int_0^\vartheta \Delta u \, dr \, d\vartheta = -\left\| \frac{\partial u}{\partial r} \right\|^2 - \left\| \frac{1}{r} \frac{\partial u}{\partial \vartheta} \right\|^2 \leq 0 \]

According to the theorem of the middle[9], we deduce that \( \lambda \left\langle r \Delta u, ru \right\rangle \leq 0 \).

We distinguish two cases:

If \( \lambda \geq 0 \) then the formula (4.4) implies that:

\[ \left\| Lu \right\|^2 \geq \left\| r \Delta u \right\|^2 \]

If \( \lambda < 0 \) we calculate \( \left\| Lu \right\|^2 = \left\| r \Delta u \right\|^2 \), it becomes:

\[ \left\| r \Delta u \right\|^2 = \left\| -r \Delta u + \lambda u - \lambda u \right\|^2 = \left\| Lu - r \lambda u \right\|^2 = \left\| Lu \right\|^2 + \lambda^2 \left\| u \right\|^2 - 2\lambda \left\langle Lu, ru \right\rangle \]
\[ \lambda \langle Lu, ru \rangle = \lambda \langle r \Delta u, ru \rangle - \lambda^2 \langle ru, ru \rangle = \lambda \langle r \Delta u, ru \rangle - \lambda^2 \| u \|^2 \]

The formula (4.5), implies that \( \lambda \langle r \Delta u, ru \rangle \geq 0 \)

Then \( \| r \Delta u \|^2 \leq \|Lu\|^2 + \lambda^2 \| u \|^2 - 2 \lambda \langle r \Delta u, ru \rangle \)

We deduce that there is a constant \( M > 0 \) such that:

\[ \| r \Delta u \|^2 \leq M \left( \|Lu\|^2 + \| u \|^2 \right) \] (4.7)

Using the formulas (4.6) and (4.7), we deduce that: there is a constant \( K > 0 \) such that:

\[ \| u \|^2_{L^2(\Omega_r)} \leq K \left( \|Lu\|^2_{L^2(\Omega_r)} + \| u \|^2_{H^1(\Omega_r)} \right) \] (4.8)

Thus according to the inequality (4.8) and the Lemma 4.1, we deduce that the image of the operator \( L \) is closed in \( L^2(\Omega_\phi) \) and the kernel of the problem (3.2) has a finite dimension, thereafter the problem (3.2) admits at least a solution in \( E^2(\Omega_\phi) \).

5. Study of the Uniqueness of the Solution.

We propose to study the uniqueness of the solution of problem (3.2). For this we calculate its kernel in space \( E^2(\Omega_\phi) \). The elements of the kernel of the problem (3.2) are solution of the following problem:

\[
\begin{cases}
\Gamma = \Omega = + \lambda \phi_\phi \text{ on } \Phi = 0 \\
\phi = \phi_\phi \text{ in } \Phi = H^1(\Omega_\phi, \Phi) \\
\int_{\Omega} \int_{\Phi} - \lambda u \phi r dr d\theta = 0 \\
\| u \|^2_{L^2(\Omega_\phi)} + \frac{1}{r} \left( \frac{\partial u}{\partial \theta} \right)^2_{L^2(\Omega_\phi)} = 0 \\
\| u \|^2_{H^1(\Omega_\phi)} + \frac{1}{r} \left( \frac{\partial u}{\partial \theta} \right)^2_{L^2(\Omega_\phi)} = 0 
\end{cases}
\] (5.1)

According to \( r \neq 0 \) in \( \Omega_\phi \), then the problem (5.1) can be written as:

\[
\begin{cases}
\lambda u = 0 \text{ in } \Omega_\phi \\
u = 0 \text{ on } \Gamma_\phi 
\end{cases}
\]

We will study the uniqueness of the solution of the problem (4.1), we have:

\[ \Delta u = 0 \Rightarrow \langle \Delta u, u \rangle = 0 \quad \langle \Delta u, u \rangle = \int_{\varphi} \int_{\Omega} - \Delta u u r d r d \theta = \int_{\varphi} \left( \frac{\partial u}{\partial r} \right)^2_{L^2(\Omega_\phi)} + \left( \frac{1}{r} \frac{\partial u}{\partial \theta} \right)^2_{L^2(\Omega_\phi)} = 0 \]

then \( u \) is a constant in \( H^1(\Omega_\phi) \), according to \( u = 0 \) on \( \Gamma_\phi \), we deduce that \( u = 0 \) in \( H^1(\Omega_\phi) \) then \( u = 0 \) in \( E^2(\Omega_\phi) \). Now we study the uniqueness of the solution of the problem (3.2), for this we calculate its kernel, we remark that:

\[ Lu = 0 \Rightarrow \langle Lu, u \rangle = \langle - \Delta u + \lambda u, u \rangle = 0 \]

we distinguish two cases.

Case1: \( \lambda > 0 \), \( \langle - \Delta u + \lambda u, u \rangle = \| \Delta u \|^2_{L^2(\Omega_\phi)} + \frac{1}{r} \left( \frac{\partial u}{\partial \theta} \right)^2_{L^2(\Omega_\phi)} + \lambda \| u \|^2_{L^2(\Omega_\phi)} = 0 \)

therefore \( u = 0 \) in \( H^1(\Omega_\phi) \) this implies that \( u = 0 \) in \( E^2(\Omega_\phi) \).

Case2 \( \lambda < 0 \) we pose \( \beta = - \lambda > 0 \), it becomes:
\[ \langle \Delta u + \beta u, u \rangle = \left\| \frac{\partial u}{\partial r} \right\|_{L^2(\Omega_\omega)} + \left\| \frac{1}{r} \frac{\partial u}{\partial \theta} \right\|_{L^2(\Omega_\omega)} = \beta \| u \|_{L^2(\Omega_\omega)} \]

then \( \| \nabla u \|_{L^2(\Omega_\omega)}^2 = \beta \| u \|_{L^2(\Omega_\omega)}^2 \) using the fact that \( \| \nabla u \|_{L^2(\Omega_\omega)}^2 \) is a norm equivalent to the usual norm of \( H^1(\Omega_\omega) \), we deduce that \( u = 0 \) in \( H^1(\Omega_\omega) \) then \( u = 0 \) in \( E^2(\Omega_\omega) \). We deduce that the problem (3.2) admits an unique solution in \( E^2(\Omega_\omega) \). This completes the proof of the theorem 4.1

6. Extension of the Study of the Problem to the Polygon.-

To extend this study to the entire polygon, we use a partition of the unit of this one. Let \( (w_k)_{k=1,N} \) is a partition of the unit of class \( C^\infty(\Omega) \) which isolates each top \( S_k; \ k = 1, N \) from \( \Omega \), \( (S_k; \ being \ the \ top \ of \ the \ angle \ \varphi_k; \ (k = 1, N) \) of \( \Omega \)

We thus have \( u = \sum_{k=1}^{N} w_k u = \sum_{k=1}^{N} u_k \).

in the neighborhood of each top \( S_k; \ k = 1, N \), the problem (3.2), becomes:

\[
\begin{cases}
  r(-\Delta u_k + \lambda u_k) = rf_k & \text{in } \Omega \\
  u_k = 0 & \text{on } \Gamma
\end{cases}
\]

If we put: \( \Psi = 1_{\Omega} - \sum_{k=1}^{N} w_k \) \( \Psi \) is a truncate function, then \( \Psi u = v \) is a function defined in an open regular domain \( O \) and thus checks the following problem:

\[
\begin{cases}
  -\Delta v + \lambda v = rf_k & \text{in } O \\
  v = 0 & \text{on } \Gamma_O
\end{cases}
\]

where \( \Gamma_O \) is the border of open regular field \( O \). For this problem it is not necessary to affect it by a weight function because it is defined on open regular domain . The study of the problem (6.1) is completely classic, then for this problem, the inequality (4.8), is checked. By adding all these inequalities and by taking the maximum we obtain an valid inequality in the polygon

References


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