

A Strong Convergence Theorem Based on a Relaxed Extragradient Method for Generalized Equilibrium Problems ¹

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Abstract

In this paper, we introduce a new relaxed extragradient viscosity approximation method for finding the common element of the set of solutions of a generalized equilibrium problems, the set of common fixed points of a finite family of nonexpansive mappings and the set solutions of the variational inequality for an inverse-strongly monotone mapping. Then, we prove a strong convergence theorem of the proposed iterative scheme in a real Hilbert spaces. Our result extend and improve many others in the literature area.

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1 Introduction

Let H be a real Hilbert space and let C be a nonempty closed convex subset of H and P_C is the metric projection of H onto C . A mapping $T : C \rightarrow C$ is called *nonexpansive* if $\|Tx - Ty\| \leq \|x - y\|$ for all $x, y \in C$. We denote $F(T) = \{x \in C : Tx = x\}$ be the set of *fixed points* of T . Recall that a self-mapping $f : C \rightarrow C$ is *contraction* if there exists a constant $\alpha \in [0, 1)$ such that $\|f(x) - f(y)\| \leq \alpha\|x - y\|$, $\forall x, y \in C$. In addition, let $A : C \rightarrow H$ be a nonlinear mapping. Let F be a bifunction of $C \times C$ into \mathbf{R} , where \mathbf{R} is the set of real numbers. The *generalized equilibrium problem* is to find $u \in C$ such that

$$F(u, v) + \langle Au, v - u \rangle \geq 0, \quad \forall v \in C. \quad (1)$$

The set of such an $u \in C$ is denoted by Ω , i.e.,

$$\Omega = \{u \in C : F(u, v) + \langle Au, v - u \rangle \geq 0, \quad \forall v \in C\}. \quad (2)$$

If $A \equiv 0$, then the problem (1) reduces to the *equilibrium problem* is to find $u \in C$ such that

$$F(u, v) \geq 0, \quad \forall v \in C. \quad (3)$$

The set of solutions of (3) is denote by $EP(F)$.

If $F \equiv 0$, then the problem (1) reduces to the *variational inequality problem* is to find $u \in C$ such that

$$\langle Au, v - u \rangle \geq 0, \quad \forall v \in C. \quad (4)$$

The set of solutions of (4) denoted by $VI(C, A)$.

The generalized equilibrium problem is very in the sense that it includes, as special cases, Numerous problems in physics, saddle point problem, fixed pint problem, variational inequality problems, optimization, economics and others. Some methods have been proposed to solve the equilibrium problem; see, for instance [4, 6, 7, 13, 19, 20]. In 1997, Combettes and Hirstoaga [3] introduced an iterative scheme of finding the best approximation to initial data when $EP(F)$ is nonempty and proved some strong convergence theorems in Hilbert spaces.

A mapping $A : C \rightarrow H$ is called *ρ -inverse-strongly monotone* (see [2, 9]) if there exists a constant $\rho > 0$ such that

$$\langle Au - Av, u - v \rangle \geq \rho\|Au - Av\|^2, \quad \forall u, v \in C.$$

For finding a common element of $F(T) \cap VI(C, A)$, Takahashi and Toyoda [17] introduced the following iterative scheme:

$$\begin{cases} x_0 = x \in C \text{ chosen arbitrary,} \\ x_{n+1} = \alpha_n x_n + (1 - \alpha_n)TP_C(x_n - \lambda_n Ax_n), \quad \forall n \geq 0, \end{cases} \quad (5)$$

where A is an ρ -inverse-strongly monotone, $\{\alpha_n\}$ is a sequence in $(0, 1)$ and $\{\lambda_n\}$ is a sequence in $(0, 2\rho)$. They showed that if $F(T) \cap VI(C, A) \neq \emptyset$, then the sequence $\{x_n\}$ generated by (5) converges weakly to $z_0 \in F(T) \cap VI(C, A)$.

For finding a common element of $F(T) \cap VI(C, A) \cap EP(F)$, Plubtieng and Punpaeng [11] introduced the following iterative scheme: $x_1 = x \in C$ and

$$\begin{cases} F(u_n, y) + \frac{1}{r_n} \langle y - u_n, u_n - x_n \rangle \geq 0, \quad \forall y \in C, \\ y_n = P_C(u_n - \lambda_n A u_n), \\ x_{n+1} = \alpha_n x + \beta_n x_n + \gamma_n T P_C(y_n - \lambda_n A y_n), \quad \forall n \geq 1. \end{cases} \tag{6}$$

They proved that under certain appropriate conditions imposed on $\{\alpha_n\}$, $\{\beta_n\}$, $\{\gamma_n\}$, $\{r_n\}$ and $\{\lambda_n\}$, the sequence $\{x_n\}$ generated by (6) converges strongly to $z_0 \in F(T) \cap VI(C, A) \cap EP(F)$, where $z_0 = P_{F(T) \cap VI(C, A) \cap EP(F)} x$.

In 2009, Kangtunyakarn and Suantai [5] introduced a mapping generated by a finite family of nonexpansive mappings as follows:

For a finite family of nonexpansive mappings T_1, T_2, \dots, T_N and sequence $\{\mu_{n,i}\}_i^N$ in $[0,1]$, they define the mapping $K_n : C \rightarrow C$ as follows:

$$\begin{aligned} U_{n,1} &= \mu_{n,1} T_1 + (1 - \mu_{n,1}) I, \\ U_{n,2} &= \mu_{n,2} T_2 U_{n,1} + (1 - \mu_{n,2}) U_{n,1}, \\ U_{n,3} &= \mu_{n,3} T_3 U_{n,2} + (1 - \mu_{n,3}) U_{n,2}, \\ &\vdots \\ U_{n,N-1} &= \mu_{n,N-1} T_{N-1} U_{n,N-2} + (1 - \mu_{n,N-1}) U_{n,N-2}, \\ K_n &= U_{n,N} = \mu_{n,N} T_N U_{n,N-1} + (1 - \mu_{n,N}) U_{n,N-1}. \end{aligned} \tag{7}$$

Let C be a nonempty convex subset of real Banach spaces. Let $\{T_i\}_{i=1}^N$ be a finite family of nonexpansive mappings of C into itself and let μ_1, \dots, μ_N be real numbers such that $0 \leq \mu_i \leq 1$ for every $i = 1, \dots, N$. They define a mapping $K : C \rightarrow C$ as follows:

$$\begin{aligned} U_1 &= \mu_1 T_1 + (1 - \mu_1) I, \\ U_2 &= \mu_2 T_2 U_1 + (1 - \mu_2) U_1, \\ U_3 &= \mu_3 T_3 U_2 + (1 - \mu_3) U_2, \\ &\vdots \\ U_{N-1} &= \mu_{N-1} T_{N-1} U_{N-2} + (1 - \mu_{N-1}) U_{N-2}, \\ K = U_N &= \mu_N T_N U_{N-1} + (1 - \mu_N) U_{N-1}. \end{aligned} \tag{8}$$

Such a mapping K is called the K -mapping generated by T_1, \dots, T_N and μ_1, \dots, μ_N . Very recently, many others studied the problems to finding a common element of the set of fixed points for a finite family of nonexpansive and the set of solutions for an generalized equilibrium problem in the setting of Hilbert spaces. (see for instance, [5, 8, 10] and the references Therein).

In this paper, motivated by the works the mentioned above, we introduced an iterative scheme (12) below for finding a common element of the set of solutions of a generalized equilibrium problem, the set of common fixed points of a finite family nonexpansive mappings and the set of solution of the variational inequality for an inverse-strongly monotone mapping in real Hilbert spaces. The results obtained in this paper improve and extend the recent ones announced many others.

2 Preliminary

Let H be a real Hilbert space with inner product $\langle \cdot, \cdot \rangle$ and norm $\| \cdot \|$, and let C be a nonempty closed convex subset of H . We denote weak convergence and strong convergence by notations \rightharpoonup and \rightarrow , respectively. For every point $x \in H$, there exists an *unique nearest point* in C , denoted by $P_C x$, such that $\|x - P_C x\| \leq \|x - y\|$ for all $y \in C$. The mapping P_C is called the *metric projection* of H onto C . It is well known that P_C is a nonexpansive mapping of H onto C and satisfies

$$\langle x - y, P_C x - P_C y \rangle \geq \|P_C x - P_C y\|^2, \quad \forall x, y \in H. \quad (9)$$

Moreover, $P_C x$ is characterized by the following properties: $P_C x \in C$ and

$$\langle x - P_C x, y - P_C x \rangle \leq 0, \quad (10)$$

$$\|x - y\|^2 \geq \|x - P_C x\|^2 + \|y - P_C x\|^2, \quad (11)$$

for all $x \in H, y \in C$. For more details see [16]. It is easy to see that the following is true:

$$u \in VI(C, A) \Leftrightarrow u = P_C(u - \lambda Au), \quad \lambda > 0.$$

A space H is said to satisfy Opial's condition [12] if for each sequence $\{x_n\}$ in H which converges weakly to point $x \in H$, we have

$$\liminf_{n \rightarrow \infty} \|x_n - x\| < \liminf_{n \rightarrow \infty} \|x_n - y\|, \quad \forall y \in H, \quad y \neq x.$$

A set-valued mapping $T : H \rightarrow 2^H$ is called *monotone* if for all $x, y \in H$, $f \in Tx$ and $g \in Ty$ imply $\langle x - y, f - g \rangle \geq 0$. A monotone mapping $T : H \rightarrow 2^H$ is *maximal* if the graph of $G(T)$ of T is not properly contained in the graph of any other monotone mapping. It is known that a monotone mapping T is maximal if and only if for $(x, f) \in H \times H$, $\langle x - y, f - g \rangle \geq 0$ for every $(y, g) \in G(T)$ implies $f \in Tx$. Let A be an ρ -inverse-strongly monotone mapping of C into H and let $N_C \varphi$ be the *normal cone* to C at $\varphi \in C$, i.e., $N_C \varphi = \{w \in H : \langle \varphi - \xi, w \rangle \geq 0, \quad \forall \xi \in C\}$ and define

$$T\varphi = \begin{cases} A\varphi + N_C \varphi, & \varphi \in C; \\ \emptyset, & \varphi \notin C. \end{cases}$$

Then T is the maximal monotone and $0 \in T\varphi$ if and only if $\varphi \in VI(C, A)$; see [14]. For solving the equilibrium problem for a bifunction $F : C \times C \rightarrow \mathbf{R}$, let us assume that F satisfies the following conditions:

- (A1) $F(x, x) = 0$ for all $x \in C$;
- (A2) F is monotone, i.e., $F(x, y) + F(y, x) \leq 0$ for all $x, y \in C$;
- (A3) for each $x, y, z \in C$, $\lim_{t \downarrow 0} F(tz + (1-t)x, y) \leq F(x, y)$;
- (A4) for each $x \in C$, $y \mapsto F(x, y)$ is convex and lower semicontinuous.

Lemma 2.1 [1]. *Let C be a nonempty closed convex subset of H and let F be a bifunction of $C \times C$ into \mathbf{R} satisfying (A1)-(A4). Let $r > 0$ and $x \in H$. Then, there exists $z \in C$ such that*

$$F(z, y) + \frac{1}{r} \langle y - z, z - x \rangle \geq 0, \quad \forall y \in C.$$

Lemma 2.2 [3]. *Assume that $F : C \times C \rightarrow \mathbf{R}$ satisfies (A1)-(A4). For $r > 0$ and $x \in H$, define a mapping $T_r : H \rightarrow C$ as follows:*

$$T_r(x) = \left\{ z \in C : F(z, y) + \frac{1}{r} \langle y - z, z - x \rangle \geq 0, \quad \forall y \in C \right\}$$

for all $z \in H$. Then, the following hold:

- (1) T_r is single-valued;
- (2) T_r is firmly nonexpansive, i.e.,

$$\|T_r x - T_r y\|^2 \leq \langle T_r x - T_r y, x - y \rangle, \quad \forall x, y \in H;$$

- (3) $F(T_r) = EP(F)$;
- (4) $EP(F)$ is closed and convex.

Lemma 2.3 [5]. *Let C be a nonempty closed convex subset of a strictly convex Banach space. Let $\{T_i\}_{i=1}^N$ be a finite family of nonexpansive mappings of C into itself with $\bigcap_{i=1}^N F(T_i) \neq \emptyset$ and $\{\mu_n\}$ be a real numbers such that $0 \leq \mu_i \leq 1$ for every $i = 1, \dots, N-1$ and $0 < \mu_N \leq 1$. Let K be the K -mapping generated by T_1, T_2, \dots, T_N and $\mu_1, \mu_2, \dots, \mu_N$. Then $F(K) = \bigcap_{i=1}^N F(T_i)$.*

Lemma 2.4 [5]. *Let C be a nonempty closed convex subset of a Banach space. Let $\{T_i\}_{i=1}^N$ be a finite family of nonexpansive mappings of C into itself and $\{\mu_{n,i}\}$ sequences in $[0, 1]$ such that $\mu_{n,i} \rightarrow \mu_n$, as $n \rightarrow \infty$, ($i=1, 2, \dots, N$). Moreover, for every $n \in N$, let K and K_n be the K -mapping generated by T_1, T_2, \dots, T_N and $\mu_1, \mu_2, \dots, \mu_N$ and T_1, T_2, \dots, T_N and $\mu_{n,1}, \mu_{n,2}, \dots, \mu_{n,N}$ respectively. Then, for every $x \in C$, we have $\lim_{n \rightarrow \infty} \|K_n x - Kx\| = 0$.*

Lemma 2.5 [5]. Let H be a Hilbert space, C a closed convex nonempty subset of H , $\{T_i\}_{i=1}^N$ be a finite family of nonexpansive mappings of C into itself with $F = \bigcap_{i=1}^N F(T_i) \neq \emptyset$, and let $F : C \times C \rightarrow \mathbf{R}$ satisfies (A1)-(A4). For every $n \in N$, let K_n be the K -mapping generated by T_1, T_2, \dots, T_N and $\mu_{n,1}, \mu_{n,2}, \dots, \mu_{n,N}$ with $\{\mu_{n,i}\}_{i=1}^N \subset [a, b]$ where $0 < a \leq b < 1$. For a sequence $\{r_n\}$ in $(0, \infty)$, let $T_{r_n} : H \rightarrow C$ be defined by

$$T_{r_n}(x) = \{z \in C : F(z, y) + \frac{1}{r} \langle y - z, z - x \rangle \geq 0, \forall y \in C\}.$$

If $\liminf_{n \rightarrow \infty} r_n > 0$, $\liminf_{n \rightarrow \infty} \frac{r_n}{r_{n+1}} = 1$ and $\lim_{n \rightarrow \infty} |\mu_{n,i} - \mu_{n-1,i}| = 0$, $\forall i \in \{1, 2, 3, \dots, N\}$, then

$$(1) \lim_{n \rightarrow \infty} \|K_{n+1}T_{r_{n+1}}w_n - K_{n+1}T_{r_n}w_n\| = 0,$$

$$(2) \lim_{n \rightarrow \infty} \|K_{n+1}w_n - K_{n+1}w_n\| = 0,$$

for every bounded sequence $\{w_n\}$ in H .

Lemma 2.6 [15]. Let $\{x_n\}$ and $\{z_n\}$ be bounded sequences in a Banach space X and let $\{\beta_n\}$ be a sequence in $[0, 1]$ with

$$0 < \liminf_{n \rightarrow \infty} \beta_n \leq \limsup_{n \rightarrow \infty} \beta_n < 1.$$

Suppose $x_{n+1} = (1 - \beta_n)z_n + \beta_n x_n$, $\forall n \geq 0$ and $\limsup_{n \rightarrow \infty} (\|z_{n+1} - z_n\| - \|x_{n+1} - x_n\|) \leq 0$. Then, $\lim_{n \rightarrow \infty} \|z_n - x_n\| = 0$.

Lemma 2.7 [18]. Assume $\{a_n\}$ is a sequence of nonnegative real numbers such that

$$a_{n+1} \leq (1 - l_n)a_n + \sigma_n, \quad n \geq 0,$$

where $\{l_n\}$ is a sequence in $(0, 1)$ and $\{\sigma_n\}$ is a sequence in \mathbf{R} such that

$$(1) \sum_{n=1}^{\infty} l_n = \infty;$$

$$(2) \limsup_{n \rightarrow \infty} \frac{\sigma_n}{l_n} \leq 0; \text{ or } \sum_{n=1}^{\infty} |\sigma_n| < \infty.$$

Then $\lim_{n \rightarrow \infty} a_n = 0$.

Lemma 2.8 Let H be a real Hilbert space. Then for all $x, y \in H$,

$$(1) \|x + y\|^2 \leq \|x\|^2 + 2\langle y, x + y \rangle;$$

$$(2) \|x + y\|^2 \geq \|x\|^2 + 2\langle y, x \rangle.$$

3 Main Results

Theorem 3.1 *Let C be a nonempty closed convex subset of a real Hilbert space H , let F be a bifunction from $C \times C$ to \mathbf{R} satisfying (A1)-(A4), let $B : C \rightarrow H$ be an ϕ -inverse-strongly monotone mapping, let $A : C \rightarrow H$ be an ρ -inverse-strongly monotone mapping and let $\{T_n\}$ be a finite family of nonexpansive mapping of C into itself such that $\Theta =: \bigcap_{n=1}^{\infty} F(T_n) \cap \Omega \cap VI(C, A) \neq \emptyset$. Let f be a contraction of H into itself with $\alpha \in [0, 1)$. Let $\{x_n\}$, $\{y_n\}$ and $\{u_n\}$ be the sequences generated by*

$$\begin{cases} x_1 = x \in C \text{ chosen arbitrary,} \\ F(u_n, y) + \langle Bx_n, y - u_n \rangle + \frac{1}{r_n} \langle y - u_n, u_n - x_n \rangle \geq 0, \quad \forall y \in C, \\ y_n = P_C(u_n - \lambda_n A u_n) \\ x_{n+1} = \alpha_n f(x_n) + \beta_n x_n + \gamma_n K_n P_C(y_n - \lambda_n A y_n), \quad \forall n \geq 1, \end{cases} \quad (12)$$

where $\{K_n\}$ is the sequence generated by (7) and $\{\alpha_n\}$, $\{\beta_n\}$, $\{\gamma_n\}$ are three sequences in $(0, 1)$, $\{\mu_{n,i}\}_{i=1}^N \subset [a, c] \subset (0, 1)$, $\lambda_n \in (0, b] \subset (0, 2\rho)$ and $r_n \in (0, d] \subset (0, 2\phi)$ satisfy the following conditions:

- (i) $\alpha_n + \beta_n + \gamma_n = 1$,
- (ii) $\lim_{n \rightarrow \infty} \alpha_n = 0$ and $\sum_{n=1}^{\infty} \alpha_n = \infty$,
- (iii) $0 < \liminf_{n \rightarrow \infty} \beta_n \leq \limsup_{n \rightarrow \infty} \beta_n < 1$,
- (iv) $\liminf_{n \rightarrow \infty} \lambda_n > 0$ and $\lim_{n \rightarrow \infty} |\lambda_{n+1} - \lambda_n| = 0$,
- (v) $\liminf_{n \rightarrow \infty} r_n > 0$ and $\lim_{n \rightarrow \infty} |r_{n+1} - r_n| = 0$,
- (vi) $\lim_{n \rightarrow \infty} |\mu_{n,i} - \mu_{n-1,i}| = 0$, for $i = 1, 2, 3, \dots, N$.

Then, $\{x_n\}$ and $\{u_n\}$ converge strongly to a point $z_0 \in \Theta$, where $z_0 = P_{\Theta} f(z_0)$.

Proof. First, we show that $I - r_n B$ is nonexpansive. Indeed, from the ϕ -inverse-strongly monotone and $r_n < 2\phi$, $\forall n \in \mathbf{N}$, we obtain

$$\begin{aligned} & \|(I - r_n B)x - (I - r_n B)y\|^2 \\ &= \|(x - y) - r_n(Bx - By)\|^2 \\ &= \|x - y\|^2 - 2r_n \langle x - y, Bx - By \rangle + r_n^2 \|Bx - By\|^2 \\ &\leq \|x - y\|^2 - 2r_n \phi \|Bx - By\|^2 + r_n^2 \|Bx - By\|^2 \\ &= \|x - y\|^2 + r_n(r_n - 2\phi) \|Bx - By\|^2 \\ &= \|x - y\|^2, \end{aligned}$$

Thus $I - \lambda_n B$ is nonexpansive. So is $I - \lambda_n A$

Next, we will divide the proof into several steps.

Step 1. We claim that $\{x_n\}$ is bounded.

By Lemma 2.2, we have $u_n = T_{r_n}(x_n - r_n Bx_n)$, $\forall n \in \mathbf{N}$. Let $z^* \in \Theta$ for each $n \geq 1$, we get $z^* = T_{r_n}(z^* - r_n Bz^*)$. By Lemma 2.2 (2), we obtain

$$\begin{aligned} \|u_n - z^*\|^2 &= \|T_{r_n}(x_n - r_n Bx_n) - T_{r_n}(z^* - r_n Bz^*)\|^2 \\ &\leq \langle u_n - z^*, (x_n - r_n Bx_n) - (z^* - r_n Bz^*) \rangle \\ &= \frac{1}{2} \left\{ \|u_n - z^*\|^2 + \|(x_n - r_n Bx_n) - (z^* - r_n Bz^*)\|^2 \right. \\ &\quad \left. - \|(u_n - z^*) - ((x_n - r_n Bx_n) - (z^* - r_n Bz^*))\|^2 \right\} \end{aligned}$$

and by $I - r_n B$ is nonexpansive implies that

$$\begin{aligned} &\|u_n - z^*\|^2 \\ &\leq \|(x_n - r_n Bx_n) - (z^* - r_n Bz^*)\|^2 - \|(u_n - x_n) - r_n(Bz^* - Bx_n)\|^2 \\ &= \|(I - r_n B)x_n - (I - r_n B)z^*\|^2 - \|(u_n - x_n) - r_n(Bz^* - Bx_n)\|^2 \\ &\leq \|x_n - z^*\|^2 - \|(u_n - x_n) - r_n(Bz^* - Bx_n)\|^2 \leq \|x_n - z^*\|^2. \end{aligned}$$

Let $z^* \in VI(C, A)$, since P_C is a nonexpansive mapping and A is an ρ -inverse-strongly monotone mapping and by (12), we obtain

$$\begin{aligned} \|y_n - z^*\|^2 &= \|P_C(u_n - \lambda_n Au_n) - P_C(z^* - \lambda_n Az^*)\|^2 \\ &\leq \|(u_n - \lambda_n Au_n) - (z^* - \lambda_n Az^*)\|^2 \\ &= \|(u_n - z^*) - \lambda_n(Au_n - Az^*)\|^2 \\ &\leq \|u_n - z^*\|^2 + \lambda_n(\lambda_n - 2\rho)\|Au_n - Az^*\|^2 \\ &\leq \|u_n - z^*\|^2 \leq \|x_n - z^*\|^2. \end{aligned}$$

Put $v_n = P_C(y_n - \lambda_n Ay_n)$. It follows that

$$\|v_n - z^*\|^2 = \|P_C(y_n - \lambda_n Ay_n) - P_C(z^* - \lambda_n Az^*)\|^2 \leq \|y_n - z^*\|^2 \leq \|x_n - z^*\|^2$$

and so

$$\begin{aligned} \|x_{n+1} - z^*\| &= \|\alpha_n f(x_n) + \beta_n x_n + \gamma_n K_n v_n - z^*\| \\ &\leq \alpha_n \|f(x_n) - z^*\| + \beta_n \|x_n - z^*\| + \gamma_n \|K_n v_n - z^*\| \\ &\leq \alpha_n \alpha \|x_n - z^*\| + \alpha_n \|f(z^*) - z^*\| + \beta_n \|x_n - z^*\| + \gamma_n \|v_n - z^*\| \\ &\leq \alpha_n \alpha \|x_n - z^*\| + \alpha_n \|f(z^*) - z^*\| + \beta_n \|x_n - z^*\| + \gamma_n \|x_n - z^*\| \\ &= [1 - \alpha_n(1 - \alpha)] \|x_n - z^*\| + \alpha_n(1 - \alpha) \frac{\|f(z^*) - z^*\|}{1 - \alpha} \\ &\leq \max \left\{ \|x_n - z^*\|, \frac{\|f(z^*) - z^*\|}{1 - \alpha} \right\} \\ &\quad \vdots \\ &\leq \max \left\{ \|x_1 - z^*\|, \frac{\|f(z^*) - z^*\|}{1 - \alpha} \right\}. \end{aligned}$$

This implies that $\{x_n\}$ is bounded, so are $\{v_n\}$, $\{y_n\}$, $\{u_n\}$ and $\{K_nv_n\}$.

Step2. We claim that $\lim_{n \rightarrow \infty} \|x_{n+1} - x_n\| = 0$.

Since $u_n = T_{r_n}(x_n - r_n Bx_n)$ and $u_{n+1} = T_{r_{n+1}}(x_{n+1} - r_{n+1} Bx_{n+1})$, we get

$$F(u_n, y) + \langle Bx_n, y - u_n \rangle + \frac{1}{r_n} \langle y - u_n, u_n - x_n \rangle \geq 0, \quad \forall y \in C \quad (13)$$

and

$$F(u_{n+1}, y) + \langle Bx_{n+1}, y - u_{n+1} \rangle + \frac{1}{r_{n+1}} \langle y - u_{n+1}, u_{n+1} - x_{n+1} \rangle \geq 0, \quad \forall y \in C. \quad (14)$$

Take $y = u_{n+1}$ in (14) and $y = u_n$ in (13), we have

$$F(u_n, u_{n+1}) + \langle Bx_n, u_{n+1} - u_n \rangle + \frac{1}{r_n} \langle u_{n+1} - u_n, u_n - x_n \rangle \geq 0$$

and

$$F(u_{n+1}, u_n) + \langle Bx_{n+1}, u_n - u_{n+1} \rangle + \frac{1}{r_{n+1}} \langle u_n - u_{n+1}, u_{n+1} - x_{n+1} \rangle \geq 0.$$

Adding the above two inequalities, the monotonicity of F implies that

$$\langle Bx_{n+1} - Bx_n, u_n - u_{n+1} \rangle + \left\langle u_n - u_{n+1}, \frac{u_{n+1} - x_{n+1}}{r_{n+1}} - \frac{u_n - x_n}{r_n} \right\rangle \geq 0$$

and so

$$\begin{aligned} 0 &\leq \left\langle u_n - u_{n+1}, r_n(Bx_{n+1} - Bx_n) + \frac{r_n}{r_{n+1}}(u_{n+1} - x_{n+1}) - (u_n - x_n) \right\rangle \\ &= \left\langle u_{n+1} - u_n, u_n - u_{n+1} + \left(1 - \frac{r_n}{r_{n+1}}\right)u_{n+1} + (x_{n+1} - r_n Bx_{n+1}) \right. \\ &\quad \left. - (x_n - r_n Bx_n) - x_{n+1} + \frac{r_n}{r_{n+1}}x_{n+1} \right\rangle \\ &= \left\langle u_{n+1} - u_n, u_n - u_{n+1} + \left(1 - \frac{r_n}{r_{n+1}}\right)(u_{n+1} - x_{n+1}) \right. \\ &\quad \left. + (x_{n+1} - r_n Bx_{n+1}) - (x_n - r_n Bx_n) \right\rangle. \end{aligned}$$

Without loss of generality, let us assume that there exists a real number c such that $r_n > c, \quad \forall n \geq 1$. Then, we have

$$\|u_{n+1} - u_n\|^2 \leq \|u_{n+1} - u_n\| \left\{ \|x_{n+1} - x_n\| + \left|1 - \frac{r_n}{r_{n+1}}\right| \|u_{n+1} - x_{n+1}\| \right\}$$

and hence

$$\begin{aligned} \|u_{n+1} - u_n\| &\leq \|x_{n+1} - x_n\| + \frac{1}{r_{n+1}}|r_{n+1} - r_n|\|u_{n+1} - x_{n+1}\| \\ &\leq \|x_{n+1} - x_n\| + \frac{1}{c}|r_{n+1} - r_n|M_1, \end{aligned} \quad (15)$$

where $M_1 = \sup\{\|u_n - x_n\| : n \in \mathbf{N}\}$. Since $I - \lambda_n A$ is nonexpansive, we have

$$\begin{aligned} &\|v_{n+1} - v_n\| \\ &= \|P_C(y_{n+1} - \lambda_{n+1}Ay_{n+1}) - P_C(y_n - \lambda_nAy_n)\| \\ &\leq \|(y_{n+1} - \lambda_{n+1}Ay_{n+1}) - (y_n - \lambda_nAy_n)\| \\ &\leq \|y_{n+1} - y_n\| + |\lambda_n - \lambda_{n+1}|\|Ay_n\| \\ &= \|P_C(u_{n+1} - \lambda_{n+1}Au_{n+1}) - P_C(u_n - \lambda_nAu_n)\| + |\lambda_n - \lambda_{n+1}|\|Ay_n\| \\ &\leq \|u_{n+1} - u_n\| + |\lambda_n - \lambda_{n+1}|\|Au_n\| + |\lambda_n - \lambda_{n+1}|\|Ay_n\| \\ &= \|u_{n+1} - u_n\| + |\lambda_n - \lambda_{n+1}|(\|Au_n\| + \|Ay_n\|). \end{aligned} \quad (16)$$

Substituting (15) into (16), we have

$$\begin{aligned} \|v_{n+1} - v_n\| &\leq \|x_{n+1} - x_n\| + \frac{M_1}{c}|r_{n+1} - r_n| \\ &\quad + |\lambda_n - \lambda_{n+1}|(\|Au_n\| + \|Ay_n\|). \end{aligned} \quad (17)$$

Setting $z_n = \frac{\alpha_n}{1-\beta_n}f(x_n) + \frac{\gamma_n}{1-\beta_n}K_nv_n$, we have $x_{n+1} = (1-\beta_n)z_n + \beta_nx_n$, $n \geq 1$. It follows that

$$\begin{aligned} &\|z_{n+1} - z_n\| \\ &= \left\| \left(\frac{\alpha_{n+1}}{1-\beta_{n+1}}f(x_{n+1}) + \frac{\gamma_{n+1}}{1-\beta_{n+1}}K_{n+1}v_{n+1} \right) - \left(\frac{\alpha_n}{1-\beta_n}f(x_n) + \frac{\gamma_n}{1-\beta_n}K_nv_n \right) \right\| \\ &= \left\| \frac{\alpha_{n+1}}{1-\beta_{n+1}}f(x_{n+1}) - \frac{\alpha_{n+1}}{1-\beta_{n+1}}f(x_n) + \frac{\alpha_{n+1}}{1-\beta_{n+1}}f(x_n) - \frac{\alpha_n}{1-\beta_n}f(x_n) \right. \\ &\quad \left. + \frac{\gamma_{n+1}}{1-\beta_{n+1}}K_{n+1}v_{n+1} - \frac{\gamma_{n+1}}{1-\beta_{n+1}}K_nv_n + \frac{\gamma_{n+1}}{1-\beta_{n+1}}K_nv_n - \frac{\gamma_n}{1-\beta_n}K_nv_n \right\| \\ &\leq \frac{\alpha_{n+1}}{1-\beta_{n+1}}\|f(x_{n+1}) - f(x_n)\| + \left| \frac{\alpha_{n+1}}{1-\beta_{n+1}} - \frac{\alpha_n}{1-\beta_n} \right| \|f(x_n)\| \\ &\quad + \frac{\gamma_{n+1}}{1-\beta_{n+1}}\|K_{n+1}v_{n+1} - K_nv_n\| + \left| \frac{\gamma_{n+1}}{1-\beta_{n+1}} - \frac{\gamma_n}{1-\beta_n} \right| \|K_nv_n\| \\ &\leq \frac{\alpha\alpha_{n+1}}{1-\beta_{n+1}}\|x_{n+1} - x_n\| + \left| \frac{\alpha_{n+1}}{1-\beta_{n+1}} - \frac{\alpha_n}{1-\beta_n} \right| (\|f(x_n)\| + \|K_nv_n\|) \\ &\quad + \left(1 - \frac{\alpha_{n+1}}{1-\beta_{n+1}} \right) \|K_{n+1}v_{n+1} - K_nv_n\| \\ &\leq \frac{\alpha\alpha_{n+1}}{1-\beta_{n+1}}\|x_{n+1} - x_n\| + \left| \frac{\alpha_{n+1}}{1-\beta_{n+1}} - \frac{\alpha_n}{1-\beta_n} \right| (\|f(x_n)\| + \|K_nv_n\|) \end{aligned}$$

$$\begin{aligned}
 & + \left(1 - \frac{\alpha_{n+1}}{1 - \beta_{n+1}}\right) \left\{ \|K_{n+1}v_{n+1} - K_{n+1}v_n\| + \|K_{n+1}v_n - K_nv_n\| \right\} \\
 \leq & \frac{\alpha\alpha_{n+1}}{1 - \beta_{n+1}} \|x_{n+1} - x_n\| + \left| \frac{\alpha_{n+1}}{1 - \beta_{n+1}} - \frac{\alpha_n}{1 - \beta_n} \right| (\|f(x_n)\| + \|K_nv_n\|) \\
 & + \left(1 - \frac{\alpha_{n+1}}{1 - \beta_{n+1}}\right) \left\{ \|v_{n+1} - v_n\| + \|K_{n+1}v_n - K_nv_n\| \right\}. \tag{18}
 \end{aligned}$$

Combining (17) and (18), we deduce

$$\begin{aligned}
 & \|z_{n+1} - z_n\| - \|x_{n+1} - x_n\| \\
 \leq & \left(\frac{\alpha\alpha_{n+1}}{1 - \beta_{n+1}} - 1 \right) \|x_{n+1} - x_n\| + \left| \frac{\alpha_{n+1}}{1 - \beta_{n+1}} - \frac{\alpha_n}{1 - \beta_n} \right| (\|f(x_n)\| + \|K_nv_n\|) \\
 & + \left(1 - \frac{\alpha_{n+1}}{1 - \beta_{n+1}}\right) \left\{ \|x_{n+1} - x_n\| + \frac{M_1}{c} |r_{n+1} - r_n| \right. \\
 & \left. + |\lambda_n - \lambda_{n+1}| (\|Au_n\| + \|Ay_n\|) + \|K_{n+1}v_n - K_nv_n\| \right\} \\
 = & \left(\frac{\alpha_{n+1}(\alpha - 1)}{1 - \beta_{n+1}} \right) \|x_{n+1} - x_n\| + \left| \frac{\alpha_{n+1}}{1 - \beta_{n+1}} - \frac{\alpha_n}{1 - \beta_n} \right| (\|f(x_n)\| + \|K_nv_n\|) \\
 & + \left(1 - \frac{\alpha_{n+1}}{1 - \beta_{n+1}}\right) \left\{ \frac{M_1}{c} |r_{n+1} - r_n| + |\lambda_n - \lambda_{n+1}| (\|Au_n\| + \|Ay_n\|) \right. \\
 & \left. + \|K_{n+1}v_n - K_nv_n\| \right\}.
 \end{aligned}$$

From conditions (i), (ii), (iv), (v) and Lemma 2.5, we obtain

$$\limsup_{n \rightarrow \infty} (\|z_{n+1} - z_n\| - \|x_{n+1} - x_n\|) \leq 0.$$

Hence, by Lemma 2.6, we obtain

$$\lim_{n \rightarrow \infty} \|z_n - x_n\| = 0.$$

It follows that

$$\lim_{n \rightarrow \infty} \|x_{n+1} - x_n\| = \lim_{n \rightarrow \infty} (1 - \beta_n) \|z_n - x_n\| = 0. \tag{19}$$

Applying (19), (iv) and (v), we have

$$\lim_{n \rightarrow \infty} \|u_{n+1} - u_n\| = \lim_{n \rightarrow \infty} \|v_{n+1} - v_n\| = 0.$$

Step 3. We claim that $\lim_{n \rightarrow \infty} \|K_nv_n - v_n\| = 0$.

Indeed, from $\|K_nv_n - v_n\| \leq \|K_nv_n - x_n\| + \|x_n - u_n\| + \|u_n - v_n\|$.

(a) First, we prove that $\|K_nv_n - x_n\| \rightarrow 0$, we note that

$$\begin{aligned}
 \|x_{n+1} - x_n\| & = \|\alpha_n f(x_n) + \beta_n x_n + \gamma_n K_nv_n - x_n\| \\
 & \leq \alpha_n \|f(x_n) - x_n\| + \gamma_n \|K_nv_n - x_n\|.
 \end{aligned}$$

Since $\alpha_n \rightarrow 0$ and $\|x_{n+1} - x_n\| \rightarrow 0$ implies that $\|K_n v_n - x_n\| \rightarrow 0$, as $n \rightarrow \infty$
 (b) Next, we prove that $\|x_n - u_n\| \rightarrow 0$, we obtain

$$\begin{aligned} \|u_n - z^*\|^2 &\leq \|x_n - z^*\|^2 - \|(u_n - x_n) + r_n(Bx_n - Bz^*)\|^2 \\ &\leq \|x_n - z^*\|^2 - \left\{ \|u_n - x_n\|^2 \right. \\ &\quad \left. + 2r_n \|u_n - x_n\| \|Bx_n - Bz^*\| + r_n^2 \|Bx_n - Bz^*\|^2 \right\} \\ &\leq \|x_n - z^*\|^2 - \|u_n - x_n\|^2. \end{aligned} \quad (20)$$

From (12), we compute

$$\begin{aligned} \|x_{n+1} - z^*\|^2 &\leq \alpha_n \|f(x_n) - z^*\|^2 + \beta_n \|x_n - z^*\|^2 + \gamma_n \|K_n v_n - z^*\|^2 \\ &\leq \alpha_n \|f(x_n) - z^*\|^2 + \beta_n \|x_n - z^*\|^2 + \gamma_n \|v_n - z^*\|^2 \\ &\leq \alpha_n \|f(x_n) - z^*\|^2 + \beta_n \|x_n - z^*\|^2 + \gamma_n \|y_n - z^*\|^2 \\ &\leq \alpha_n \|f(x_n) - z^*\|^2 + \beta_n \|x_n - z^*\|^2 + \gamma_n \|u_n - z^*\|^2. \end{aligned} \quad (21)$$

Substituting (20) into (21), we obtain

$$\begin{aligned} &\|x_{n+1} - z^*\|^2 \\ &\leq \alpha_n \|f(x_n) - z^*\|^2 + \beta_n \|x_n - z^*\|^2 + \gamma_n \left\{ \|x_n - z^*\|^2 - \|u_n - x_n\|^2 \right\} \\ &= \alpha_n \|f(x_n) - z^*\|^2 + (1 - \alpha_n) \|x_n - z^*\|^2 - \gamma_n \|u_n - x_n\|^2 \\ &\leq \alpha_n \|f(x_n) - z^*\|^2 + \|x_n - z^*\|^2 - \gamma_n \|u_n - x_n\|^2. \end{aligned}$$

It follows that

$$\begin{aligned} \gamma_n \|u_n - x_n\|^2 &\leq \alpha_n \|f(x_n) - z^*\|^2 + \|x_n - z^*\|^2 - \|x_{n+1} - z^*\|^2, \\ &\leq \alpha_n \|f(x_n) - z^*\|^2 + \|x_n - x_{n+1}\| (\|x_n - z^*\| + \|x_{n+1} - z^*\|). \end{aligned}$$

By virtue of conditions (i)-(iii), $\liminf_{n \rightarrow \infty} r_n > 0$ and (19), we have

$$\lim_{n \rightarrow \infty} \|x_n - u_n\| = 0.$$

(c) Now, we prove that $\|v_n - u_n\| \rightarrow 0$. We observe that

$$\begin{aligned} \|y_n - z^*\|^2 &\leq \|u_n - z^*\|^2 + \lambda_n (\lambda_n - 2\rho) \|Au_n - Az^*\|^2, \\ &\leq \|x_n - z^*\|^2 + a(b - 2\rho) \|Au_n - Az^*\|^2. \end{aligned} \quad (22)$$

Substituting (22) into (21), we have

$$\begin{aligned} \|x_{n+1} - z^*\|^2 &\leq \alpha_n \|f(x_n) - z^*\|^2 + \beta_n \|x_n - z^*\|^2 \\ &\quad + \gamma_n \left\{ \|x_n - z^*\|^2 + a(b - 2\rho) \|Au_n - Az^*\|^2 \right\} \\ &= \alpha_n \|f(x_n) - z^*\|^2 + (1 - \alpha_n) \|x_n - z^*\|^2 \\ &\quad + \gamma_n a(b - 2\rho) \|Au_n - Az^*\|^2 \\ &\leq \alpha_n \|f(x_n) - z^*\|^2 + \|x_n - z^*\|^2 + \gamma_n a(b - 2\rho) \|Au_n - Az^*\|^2 \end{aligned}$$

and hence

$$\begin{aligned} & \gamma_n a(2\rho - b) \|Au_n - Az^*\|^2 \\ \leq & \alpha_n \|f(x_n) - z^*\|^2 + \|x_{n+1} - x_n\|(\|x_n - z^*\| + \|x_{n+1} - z^*\|). \end{aligned}$$

Since $\alpha_n \rightarrow 0$, $a, b \in (0, 2\rho)$, $\liminf_{n \rightarrow \infty} \gamma_n > 0$ and $\|x_{n+1} - x_n\| \rightarrow 0$, we obtain $\|Au_n - Az^*\| \rightarrow 0$, as $n \rightarrow \infty$. We compute

$$\begin{aligned} \|y_n - z^*\|^2 &= \|P_C(u_n - \lambda_n Au_n) - P_C(z^* - \lambda_n Az^*)\|^2 \\ &\leq \langle (u_n - \lambda_n Au_n) - (z^* - \lambda_n Az^*), y_n - z^* \rangle \\ &= \frac{1}{2} \left\{ \|(u_n - \lambda_n Au_n) - (z^* - \lambda_n Az^*)\|^2 + \|y_n - z^*\|^2 \right. \\ &\quad \left. - \|(u_n - \lambda_n Au_n) - (z^* - \lambda_n Az^*) - (y_n - z^*)\|^2 \right\} \\ &\leq \frac{1}{2} \left\{ \|u_n - z^*\|^2 + \|y_n - z^*\|^2 - \|(u_n - y_n) - \lambda_n (Au_n - Az^*)\|^2 \right\} \\ &\leq \frac{1}{2} \left\{ \|u_n - z^*\|^2 + \|y_n - z^*\|^2 - \|u_n - y_n\|^2 \right. \\ &\quad \left. + 2\lambda_n \langle u_n - y_n, Au_n - Az^* \rangle - \lambda_n^2 \|Au_n - Az^*\|^2 \right\}, \end{aligned}$$

this implies that

$$\begin{aligned} \|y_n - z^*\|^2 &\leq \|u_n - z^*\|^2 - \|u_n - y_n\|^2 + 2\lambda_n \langle u_n - y_n, Au_n - Az^* \rangle \\ &\leq \|x_n - z^*\|^2 - \|u_n - y_n\|^2 + 2\lambda_n \langle u_n - y_n, Au_n - Az^* \rangle. \end{aligned} \quad (23)$$

Substituting (23) into (21), we have

$$\begin{aligned} \|x_{n+1} - z^*\|^2 &\leq \alpha_n \|f(x_n) - z^*\|^2 + \beta_n \|x_n - z^*\|^2 + \gamma_n \left\{ \|x_n - z^*\|^2 \right. \\ &\quad \left. - \|u_n - y_n\|^2 + 2\lambda_n \langle u_n - y_n, Au_n - Az^* \rangle \right\} \\ &\leq \alpha_n \|f(x_n) - z^*\|^2 + \|x_n - z^*\|^2 - \gamma_n \|u_n - y_n\|^2 \\ &\quad + 2\lambda_n \gamma_n \langle u_n - y_n, Au_n - Az^* \rangle, \end{aligned}$$

which implies that

$$\begin{aligned} \gamma_n \|u_n - y_n\|^2 &\leq \alpha_n \|f(x_n) - z^*\|^2 + \|x_n - z^*\|^2 - \|x_{n+1} - z^*\|^2 \\ &\quad + 2\lambda_n \gamma_n \|u_n - y_n\| \|Au_n - Az^*\| \\ &\leq \alpha_n \|f(x_n) - z^*\|^2 + \|x_{n+1} - x_n\|(\|x_n - z^*\| + \|x_{n+1} - z^*\|) \\ &\quad + 2\lambda_n \gamma_n \|u_n - y_n\| \|Au_n - Az^*\|. \end{aligned}$$

By conditions, (i)-(iii), $\|x_{n+1} - x_n\| \rightarrow 0$ and $\|Au_n - Az^*\| \rightarrow 0$, we obtain $\|u_n - y_n\| \rightarrow 0$, as $n \rightarrow \infty$.

We note that

$$\begin{aligned} \|v_n - u_n\| &\leq \|v_n - y_n\| + \|y_n - u_n\| \\ &= \|P_C(y_n - \lambda_n Ay_n) - P_C(u_n - \lambda_n Au_n)\| + \|y_n - u_n\| \\ &\leq \|y_n - u_n\| + \|y_n - u_n\| = 2\|y_n - u_n\| \rightarrow 0, \text{ as } n \rightarrow \infty. \end{aligned}$$

Hence

$$\|K_nv_n - v_n\| \leq \|K_nv_n - x_n\| + \|x_n - u_n\| + \|u_n - v_n\| \rightarrow 0, \text{ as } n \rightarrow \infty. \quad (24)$$

Let K be the mapping defined by (8). Since $\{v_n\}$ is bounded, applying Lemma 2.4 and (24), we have

$$\|Kv_n - v_n\| \leq \|Kv_n - K_nv_n\| + \|K_nv_n - v_n\| \rightarrow 0 \text{ as } n \rightarrow \infty.$$

Step 4. We claim that $\limsup_{n \rightarrow \infty} \langle f(z_0) - z_0, x_n - z_0 \rangle \leq 0$, where $z_0 = P_\Theta f(z_0)$.

To show this inequality, we choose a subsequence $\{v_{n_i}\}$ of $\{v_n\}$ such that

$$\lim_{i \rightarrow \infty} \langle f(z_0) - z_0, K_nv_{n_i} - v_{n_i} \rangle = \limsup_{n \rightarrow \infty} \langle f(z_0) - z_0, K_nv_n - v_n \rangle.$$

Since $\{v_{n_i}\}$ is bounded, there exists a subsequence $\{v_{n_{i_j}}\}$ of $\{v_{n_i}\}$ which converges weakly to $z \in C$. Without loss of generality, we can assume that $v_{n_{i_j}} \rightharpoonup z$. From $\|K_nv_n - v_n\| \rightarrow 0$, we obtain $K_nv_{n_i} \rightharpoonup z$.

By Theorem 3.1 in [10] pp.14-17, we can prove that $z \in \Theta$. Thus, by the property of the metric projection implies that

$$\begin{aligned} \limsup_{n \rightarrow \infty} \langle f(z_0) - z_0, x_n - z_0 \rangle &= \limsup_{n \rightarrow \infty} \langle f(z_0) - z_0, x_n - K_nv_n + K_nv_n - z_0 \rangle \\ &\leq \limsup_{n \rightarrow \infty} \langle f(z_0) - z_0, K_nv_n - z_0 \rangle \\ &= \limsup_{n \rightarrow \infty} \langle f(z_0) - z_0, K_{n_i}v_{n_i} - z_0 \rangle \\ &= \limsup_{n \rightarrow \infty} \langle f(z_0) - z_0, z - z_0 \rangle \leq 0. \end{aligned} \quad (25)$$

From Step 3(a), we have $\|K_nv_n - x_n\| \rightarrow 0$, then

$$\begin{aligned} \limsup_{n \rightarrow \infty} \langle f(z_0) - z_0, x_{n+1} - z_0 \rangle &\leq \limsup_{n \rightarrow \infty} \langle f(z_0) - z_0, x_{n+1} - x_n \rangle \\ &\quad + \limsup_{n \rightarrow \infty} \langle f(z_0) - z_0, x_n - z_0 \rangle. \end{aligned}$$

Since $\|x_{n+1} - x_n\| \rightarrow 0$ and (25), we obtain

$$\limsup_{n \rightarrow \infty} \langle f(z_0) - z_0, x_{n+1} - z_0 \rangle \leq 0. \quad (26)$$

Step 5. Finally, we show that $x_n \rightarrow z_0$ as $n \rightarrow \infty$. We observe that

$$\begin{aligned} &\|x_{n+1} - z_0\|^2 \\ &= \langle \alpha_n f(x_n) + \beta_n x_n + \gamma_n K_nv_n - z_0, x_{n+1} - z_0 \rangle \\ &\leq \alpha_n \langle f(x_n) - f(z_0), x_{n+1} - z_0 \rangle + \alpha_n \langle f(z_0) - z_0, x_{n+1} - z_0 \rangle \\ &\quad + \frac{1}{2} \beta_n (\|x_n - z_0\|^2 + \|x_{n+1} - z_0\|^2) + \frac{1}{2} \gamma_n (\|K_nv_n - z_0\|^2 + \|x_{n+1} - z_0\|^2) \end{aligned}$$

$$\begin{aligned}
 &\leq \alpha_n \alpha^2 \langle x_n - z_0, x_{n+1} - z_0 \rangle + \alpha_n \langle f(z_0) - z_0, x_{n+1} - z_0 \rangle \\
 &\quad + \frac{1}{2} \beta_n (\|x_n - z_0\|^2 + \|x_{n+1} - z_0\|^2) + \frac{1}{2} \gamma_n (\|v_n - z_0\|^2 + \|x_{n+1} - z_0\|^2) \\
 &\leq \frac{1}{2} \alpha_n \alpha^2 (\|x_n - z_0\|^2 + \|x_{n+1} - z_0\|^2) + \alpha_n \langle f(z_0) - z_0, x_{n+1} - z_0 \rangle \\
 &\quad + \frac{1}{2} \beta_n (\|x_n - z_0\|^2 + \|x_{n+1} - z_0\|^2) + \frac{1}{2} \gamma_n (\|x_n - z_0\|^2 + \|x_{n+1} - z_0\|^2) \\
 &\leq \frac{1}{2} \alpha_n \alpha^2 (\|x_n - z_0\|^2 + \|x_{n+1} - z_0\|^2) + \alpha_n \langle f(z_0) - z_0, x_{n+1} - z_0 \rangle \\
 &\quad + \frac{1}{2} (1 - \alpha_n) (\|x_n - z_0\|^2 + \|x_{n+1} - z_0\|^2) \\
 &\leq \frac{1}{2} [1 - (1 - \alpha^2) \alpha_n] \|x_n - z_0\|^2 + \frac{1}{2} \|x_{n+1} - z_0\|^2 + 2\alpha_n \langle f(z_0) - z_0, x_{n+1} - z_0 \rangle.
 \end{aligned}$$

Which implies that

$$\|x_{n+1} - z_0\|^2 \leq \frac{1}{2} [1 - (1 - \alpha^2) \alpha_n] \|x_n - z_0\|^2 + \alpha_n \langle f(z_0) - z_0, x_{n+1} - z_0 \rangle.$$

Taking $a_n = \|x_n - z_0\|^2$, $l_n = (1 - \alpha^2) \alpha_n$ and $\sigma_n = 2\alpha_n \langle f(z_0) - z_0, x_{n+1} - z_0 \rangle$. By (26), we have $\limsup \frac{\sigma_n}{l_n} \leq 0$. It follow from Lemma 2.7 that $x_n \rightarrow z_0$. Consequently from $\|x_n - u_n\| \rightarrow 0$, we obtain $u_n \rightarrow z_0$. This complete the proof.

Corollary 3.2 *Let C be a nonempty closed convex subset of a real Hilbert space H , let F be a bifunction from $C \times C$ to \mathbf{R} satisfying (A1)-(A4), let $B : C \rightarrow H$ be an ϕ -inverse-strongly monotone mapping and let $\{T_n\}$ be a finite family of nonexpansive mapping of C into itself such that $\Theta =: \bigcap_{n=1}^{\infty} F(T_n) \cap \Omega \neq \emptyset$. Let f be a contraction of H into itself with $\alpha \in (0, 1)$. Let $\{x_n\}$, $\{y_n\}$ and $\{u_n\}$ be the sequences generated by*

$$\begin{cases} x_1 = x \in C \text{ chosen arbitrary,} \\ F(u_n, y) + \langle Bx_n, y - u_n \rangle + \frac{1}{r_n} \langle y - u_n, u_n - x_n \rangle \geq 0, \quad \forall y \in C, \\ x_{n+1} = \alpha_n f(x_n) + \beta_n x_n + \gamma_n K_n u_n, \quad \forall n \geq 1, \end{cases}$$

where $\{K_n\}$ is the sequence generated by (8) and $\{\alpha_n\}$, $\{\beta_n\}$, $\{\gamma_n\}$ are three sequences in $(0, 1)$, $\{\mu_{n,i}\}_{i=1}^N \subset [a, c] \subset (0, 1)$ and $r_n \in (0, d] \subset (0, 2\phi)$ satisfy the following conditions:

- (i) $\alpha_n + \beta_n + \gamma_n = 1$,
- (ii) $\lim_{n \rightarrow \infty} \alpha_n = 0$ and $\sum_{n=1}^{\infty} \alpha_n = \infty$,
- (iii) $0 < \liminf_{n \rightarrow \infty} \beta_n \leq \limsup_{n \rightarrow \infty} \beta_n < 1$,
- (iv) $\liminf_{n \rightarrow \infty} r_n > 0$ and $\lim_{n \rightarrow \infty} |r_{n+1} - r_n| = 0$,

$$(v) \lim_{n \rightarrow \infty} |\mu_{n,i} - \mu_{n-1,i}| = 0, \text{ for } i = 1, 2, 3, \dots, N.$$

Then, $\{x_n\}$ and $\{u_n\}$ converge strongly to a point $z_0 \in \Theta$, where $z_0 = P_{\Theta}f(z_0)$.

Proof. Put $A = 0$ in Theorem 3.1. Then $y_n = u_n$. The conclusion of Corollary 3.2 can be obtained the desired result easily.

Corollary 3.3 *Let C be a nonempty closed convex subset of a real Hilbert space H , let F be a bifunction from $C \times C$ to \mathbf{R} satisfying (A1)-(A4) and let $\{T_n\}$ be a finite family of nonexpansive mapping of C into itself such that $\Theta =: \bigcap_{n=1}^{\infty} F(T_n) \cap EP(F) \neq \emptyset$. Let f be a contraction of H into itself with $\alpha \in (0, 1)$. Let $\{x_n\}$, $\{y_n\}$ and $\{u_n\}$ be the sequences generated by*

$$\begin{cases} x_1 = x \in C \text{ chosen arbitrary,} \\ F(u_n, y) + \frac{1}{r_n} \langle y - u_n, u_n - x_n \rangle \geq 0, \quad \forall y \in C, \\ x_{n+1} = \alpha_n f(x_n) + \beta_n x_n + \gamma_n K_n u_n, \quad \forall n \geq 1, \end{cases}$$

where $\{K_n\}$ is the sequence generated by (8) and $\{\alpha_n\}$, $\{\beta_n\}$, $\{\gamma_n\}$ are three sequences in $(0, 1)$, $\{\mu_{n,i}\}_{i=1}^N \subset [a, c] \subset (0, 1)$ and $r_n \in (0, d] \subset (0, \infty)$ satisfy the following conditions:

$$(i) \alpha_n + \beta_n + \gamma_n = 1,$$

$$(ii) \lim_{n \rightarrow \infty} \alpha_n = 0 \text{ and } \sum_{n=1}^{\infty} \alpha_n = \infty,$$

$$(iii) 0 < \liminf_{n \rightarrow \infty} \beta_n \leq \limsup_{n \rightarrow \infty} \beta_n < 1,$$

$$(iv) \liminf_{n \rightarrow \infty} r_n > 0 \text{ and } \lim_{n \rightarrow \infty} |r_{n+1} - r_n| = 0,$$

$$(v) \lim_{n \rightarrow \infty} |\mu_{n,i} - \mu_{n-1,i}| = 0, \text{ for } i = 1, 2, 3, \dots, N.$$

Then, $\{x_n\}$ and $\{u_n\}$ converge strongly to a point $z_0 \in \Theta$, where $z_0 = P_{\Theta}f(z_0)$.

Proof. Put $A = B = 0$ in Theorem 3.1. Then $y_n = u_n$ and for any $\phi > 0$, we see that $\langle Bx - By, x - y \rangle \geq \phi \|Bx - By\|^2$, $\forall x, y \in C$. Let $\{r_n\}$ be a sequence satisfying the restriction: $0 \leq r_n \leq d$, where $d \in (0, \infty)$. Then we can obtain the desired conclusion easily from theorem 3.1.

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References

- [1] E. Blum and W. Oettli, From optimization and variational inequalities to equilibrium problems, *The Mathematics Student*, **63**(1994), 123–145.
- [2] F.E. Browder and W.V. Petryshyn, Construction of fixed points of non-linear mappings in Hilbert space, *Journal of Mathematical Analysis and Applications*, **20**(1967), 197–228.
- [3] P.L. Combettes and S.A. Hirstoaga, Equilibrium programming using proximal-like algorithms, *Mathematics Programming*, **78**(1997), 29–41.
- [4] V. Colao, G. Marino and H.K. Xu, An iterative method for finding common solutions of equilibrium and fixed point problems, *Journal of Mathematical Analysis and Applications*, **344**(2008), 340–352.
- [5] A. Kangtunyakarn and S. Suantai, A new mapping for finding common solutions of equilibrium problems and fixed point problems of finite family of nonexpansive mappings, *Nonlinear Analysis*, **71**(2009), 4448–4446.
- [6] C. Jaiboon, and P. Kumam, A hybrid extragradient viscosity approximation method for solving equilibrium problems and fixed point problems of infinitely many nonexpansive mappings, *Fixed Point Theory and Applications*, Article ID 374815 (2009), 32 pages.
- [7] C. Jaiboon, P. Kumam and H.W. Humphries, Weak Convergence Theorem by an Extragradient Method for Variational Inequality, Equilibrium and Fixed Point Problems, *Bulletin of Malaysian Mathematical Sciences Society*, (2)**32**(2)(2009), 173–185.
- [8] C. Jaiboon, W. Chantarangsi and P. Kumam, A convergence theorem based on a hybrid relaxed extragradient method for generalized equilibrium problems and fixed point problems of a finite family of nonexpansive mappings, *Nonlinear Anal: Hybrid Systems* (2009), doi:10.1016/j.nahs.2009.09.009.
- [9] F. Liu and M.Z. Nashed, Regularization of nonlinear Ill-posed variational inequalities and convergence rates, *Set-Valued Analysis*, **6**(1998), 313–344.
- [10] Q. Y. Liu, W.U. Zeng and N. J. Huang, An Iterative Method for Generalized Equilibrium Problems, Fixed Point Problems and Variational Inequality Problems, *Fixed Point Theory and Applications*, Article ID 531308 (2009), 20 pages.
- [11] S. Plubtieng and R. Punpaenga, A general iterative method for equilibrium problems and fixed point problems in Hilbert spaces, *Journal of Mathematical Analysis and Applications*, **336**(2007), 455–469.

- [12] Z. Opial, Weak convergence of the sequence of successive approximations for nonexpansive mappings, *Bulletin of the American Mathematical Society*, **73**(1976), 595–597.
- [13] X. Qin, M. Shang and Y. Su, A general iterative method for equilibrium problems and fixed point problems in Hilbert spaces, *Nonlinear Analysis*, **69**(2008) 38973909
- [14] R.T. Rockafellar, On the maximality of sums of nonlinear monotone operators, *Transactions of the American Mathematical Society*, **149**(1970), 75–88.
- [15] T. Suzuki, Strong convergence of krasnoselskii and Mann’s type sequences for one-parameter nonexpansive semigroups without bochner integrals, *Journal of Mathematical Analysis and Applications*, **305**(2005), 227–239.
- [16] W. Takahashi, Nonlinear functional analysis, Yokohama Publishers, Yokohama, 2000.
- [17] W. Takahashi and M. Toyoda, Weak convergence theorems for nonexpansive mappings and monotone mappings, *Journal of Optimization Theory and Applications*, **118**(2003), 417–428.
- [18] H.K. Xu, Viscosity approximation methods for nonexpansive mappings, *Journal of Mathematical Analysis and Applications*, **298**(2004), 279–291.
- [19] Y. Yao, Y.C. Liou and J.C. Yao, Convergence theorem for equilibrium problems and fixed point problems of infinite family of nonexpansive mappings, *Fixed Point Theory and Applications*, Article ID 64363 (2007), 12 pages.
- [20] S.S. Chang, H.W. Joseph Lee and C.K. Chan, A new method for solving equilibrium problem fixed point problem and variational inequality problem with application to optimization, *Nonlinear Analysis*, **70**(2009), 3307–3319.

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