Observer for Perturbed Linear Continuous Systems

Mostafa RACHIK, Hassan LAARABI, Ouafa EL KAHLAOUI and Smahane SAADI

Department of Mathematics and Computer Sciences
Faculty of Sciences Ben M’sik
Bd Comandant Idriss El harti, 20450, Casablanca, Morocco

Abstract

This work studies the construction of observers for a class of finite dimensional systems in which the dynamics are partially unknown. In this work we consider the class of the dynamics with a single unknown element in the first one and in the second one the case of several unknown elements. We give sufficient conditions in which the existence of an observer is assured. We illustrate the results by some numerical examples.

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1 Introduction

The observer design problem for linear finite dimensional systems, has attracted the interest of many researchers, with the objective to estimate the state of the system with a dynamic matrix completely known and an unknown initial state. Important results on the observer design problem in this case can be found in (Luenberger, 1966; Kalman; Edwards and Spurgeon, 1998; Ha et al., 2003; Hui and Zak, 1990; Koshkouei and Zinober, 2004; Utkin et al., 1999; Walcott and Zak, 1987; 1988; Walcott et al., 1987;...). In practice, The exact and complete knowledge of the dynamics of the system is generally impossible. Then it is very important to take into account the uncertainties that affect the dynamics of the system. In this paper, the focus will be on the observer design problem, for linear finite dimensional systems, with unknown dynamics. We
are going to study four cases according to the position of the unknown elements of the dynamics. In case one, there will be only one unknown element of the dynamics. In the second case, one whole column of the dynamics will be unknown. In the third case, the diagonal elements of the dynamics will be unknown. Finally, in the fourth case, there will be several unknown elements of the dynamics. In all cases we give a sufficient and necessary condition on the known part of the dynamics, in which our objective is achievable.

2 Problem formulation

Let us consider the linear continuous time-invariant system given by

\[
\begin{align*}
(S) \quad \begin{cases} 
\dot{x}(t) &= Ax(t) \\
y(t) &= Cx(t)
\end{cases}
\end{align*}
\]

where \( x(t) \in \mathbb{R}^n \) is the state and \( y(t) \in \mathbb{R}^p \) is the measured output, \( A \in \mathcal{M}_n(\mathbb{R}) \) and \( C \in \mathcal{M}_{p,n}(\mathbb{R}) \) are real constant matrices. It is assumed that the dynamics of the system, is not well known. The objective is the conception of Luenberger observer to estimate the state of the system. The matrix \( A \) can be represented as \( A = A + F \), with \( A \) matrix supposed to be known and \( F \) the unknown part of dynamic.

3 Main Results

3.1 Case 1: Dynamic with a single unknown element

We suppose that \( A = A + \alpha F \), where: \( |\alpha| < 1 \),
\[
A = \begin{pmatrix} 0 & \times & \cdots & \times \\ \times & \times & \cdots & \times \\ \vdots & \vdots & \ddots & \vdots \\ \times & \times & \cdots & \times \end{pmatrix} \quad \text{and} \quad F = \begin{pmatrix} 1 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 \end{pmatrix}
\]
so, the system (S) can be written
\[
\begin{align*}
\dot{x}(t) &= Ax(t) + \alpha Fx(t) \\
y(t) &= Cx(t)
\end{align*}
\]

Our aim is to design a state observer described by
\[
(Obs) \quad \begin{cases} 
\dot{z}(t) &= Az(t) + L(y(t) - \hat{y}(t)) \\
\hat{y}(t) &= Cz(t)
\end{cases}
\]
with $A - LC$ assumed to be stable.

Once introduced the error vector

$$e(t) = z(t) - Tx(t) \quad (4)$$

It is a matter of simple computations to show that the error vector updates according to the following equation

$$\dot{e}(t) = (A - LC)e(t) + \left[(A - LC)T + LC - TA - \alpha TF\right]x(t) \quad (5)$$

We introduced a new variable

$$X(t) = \begin{bmatrix} e(t) \\ x(t) \end{bmatrix} \quad (6)$$

Then (5) becomes

$$\dot{X}(t) = \begin{bmatrix} A - LC & (A - LC)T + LC - TA \\ 0 & A - LC \end{bmatrix} X(t)$$

$$+ \begin{bmatrix} 0 & -\alpha TF \\ 0 & LC + \alpha F \end{bmatrix} X(t) \quad (7)$$

we can rewrite the uncertainty matrix $\begin{bmatrix} 0 & -\alpha TF \\ 0 & LC + \alpha F \end{bmatrix}$ as follows

$$\begin{bmatrix} 0 & -\alpha TF \\ 0 & LC + \alpha F \end{bmatrix} = \begin{bmatrix} 0 & -T \\ LC & I \end{bmatrix} \begin{bmatrix} I & 0 \\ 0 & \alpha F \end{bmatrix} \begin{bmatrix} 0 & I \\ 0 & I \end{bmatrix} \quad (8)$$

which gives

$$\dot{X}(t) = \overline{A}X(t) + D\Delta EX(t) \quad (9)$$

where

$$\overline{A} = \begin{bmatrix} A - LC & (A - LC)T + LC - TA \\ 0 & A - LC \end{bmatrix}$$

Since $|\alpha| < 1$ it is obvious that $\Delta'\Delta \leq I$, then we use the following result to establish the efficiency of our observer.

**Theorem 3.1** [4] Under the hypothesis $|\alpha| < 1$, we have $\Delta'\Delta \leq I$, consequently the System (9) is quadratically stable if and only if the LMI

$$\overline{A}P + P\overline{A} + PDD'P + E'E < 0$$

has a symmetric positive defined solution $P$. 
Example 3.2 Consider the uncertain system with parameters as follows

\[ A = \begin{pmatrix} \alpha & -25 \\ 30 & -60 \end{pmatrix}, \quad L = \begin{bmatrix} 0.5 & -1 \end{bmatrix}', \quad C = \begin{bmatrix} 1.6 & -10 \end{bmatrix} \quad \text{and} \quad T = \begin{pmatrix} -3 & 0 \\ 0 & -1 \end{pmatrix}. \]

We use the Matlab LMI Control Toolbox, and we obtain the solution as follows

\[ P = \begin{bmatrix} 0.4886 & -0.5547 & -0.5587 & -0.1420 \\ -0.5547 & 1.2668 & 1.0028 & 0.3462 \\ -0.5587 & 1.0028 & 1.1115 & 0.0324 \\ -0.1420 & 0.3462 & 0.0324 & 0.7891 \end{bmatrix} \]

![Figure 1: error For \( \alpha = 0.3 \)](image)

Example 3.3 Consider the uncertain system with parameters as follows

\[ A = \begin{pmatrix} \alpha & -1 \\ 3 & -0.5 \end{pmatrix}, \quad L = \begin{bmatrix} 0.5 & 2 \end{bmatrix}', \quad C = \begin{bmatrix} 1.6 & 1 \end{bmatrix} \quad \text{and} \quad T = I. \]

We use the Matlab LMI Control Toolbox, and we obtain

\[ P = \begin{bmatrix} 2.0500 & -0.5039 & 0.6836 & -0.0948 \\ -0.5039 & 2.5258 & -0.2797 & 0.2485 \\ 0.6836 & -0.2797 & 0.6686 & -0.0473 \\ -0.0948 & 0.2485 & -0.0473 & 0.5586 \end{bmatrix} \]
3.2 Case 2: Dynamic with an unknown column

In this case, we suppose that $A = A + F$,

$$A = \begin{pmatrix} 0 & \cdots & 0 \\ 0 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & 0 \end{pmatrix} \quad \text{and} \quad F = \begin{pmatrix} a_1 & 0 & \cdots & 0 \\ a_2 & 0 & \cdots & 0 \\ \vdots & \ddots & \vdots & \vdots \\ a_n & 0 & \cdots & 0 \end{pmatrix}$$

We follow the same approach as in the first case, we obtain:

$$\dot{X}(t) = AX(t) + D\Delta EX(t) \quad (10)$$

The condition $\Delta'\Delta \leq I$ is satisfied if and only if $1 - \sum_{i=1}^{i=n} a_i^2 \geq 0$, consequently if $1 - \sum_{i=1}^{i=n} a_i^2 \geq 0$, the system (10) is quadratically stable if and only if the LMI

$$\bar{A}P + PA + PDD'P + E'E < 0$$

has a symmetric positive defined solution $P$.

**Example 3.4** Consider the uncertain system with parameters as follows

$$A = \begin{pmatrix} \alpha \\ \beta \\ -25 \\ -6 \end{pmatrix}, \quad L = [0.5 \quad -1]'$$

$$C = [20 \quad -10], \quad T = \begin{pmatrix} -3 \\ 0 \quad 2 \end{pmatrix}$$

We use the Matlab LMI Control Toolbox, and we obtain the solution

$$P = \begin{bmatrix} 0.0568 & 0.0173 & -0.2965 & -0.1290 \\ 0.0173 & 0.1047 & -0.0816 & -0.1125 \\ -0.2965 & -0.0816 & 1.5730 & 0.6994 \\ -0.1290 & -0.1125 & 0.6994 & 0.4329 \end{bmatrix}$$
3.3 Case 3: Dynamic with unknown diagonal

We suppose that $A = A + F$, where $A = \begin{pmatrix} 0 & \cdots & \times \\ \times & \cdots & \times \\ \vdots & \ddots & \vdots \\ \times & \cdots & 0 \end{pmatrix}$

and $F = \begin{pmatrix} a_1 & 0 & \cdots & 0 \\ 0 & a_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & a_n \end{pmatrix}$

As, for previous cases we have

$$\dot{X}(t) = \overline{A}X(t) + D\Delta EX(t) \quad (11)$$

The condition $\Delta'\Delta \leq I$ is satisfied if and only if $1 - a_i^2 \geq 0$ for each $i = 1, \ldots, n$, consequently if $1 - a_i^2 \geq 0$ for each $i = 1, \ldots, n$, the system (11) is quadratically stable if and only if the LMI

$$\overline{A}P + P \overline{A} + PDD'P + E'E < 0$$

has a symmetric positive defined solution $P$.

**Example 3.5** Consider the uncertain system given by

$$A = \begin{pmatrix} \alpha & 0 \\ 2 & -1 \end{pmatrix}, \quad L = [3 \ 0.2]', \quad C = [1 \ 0.2], \quad T = \begin{pmatrix} 2 & 1 \\ 0 & -1 \end{pmatrix}$$
Figure 3: error for $\alpha = -0.3$ and $\beta = 0.2$

We use the Matlab LMI Control Toolbox, and we obtain the solution as follows:

$$P = \begin{bmatrix} 0.5527 & 0.4438 & 0.4467 & -0.3108 \\ 0.4438 & 1.1836 & -0.0273 & -1.7325 \\ 0.4467 & -0.0273 & 0.9009 & 0.3897 \\ -0.3108 & -1.7325 & 0.3897 & 3.4638 \end{bmatrix}$$

3.4 Case 4: Dynamic with several unknown elements

$$A = A + F,$$ where $A = \begin{pmatrix} 0 & \times & \cdots & \times \\ \times & \times & \cdots & 0 \\ \vdots & \times & \ddots & \vdots \\ \times & 0 & \cdots & \times \end{pmatrix}$ and $F = \begin{pmatrix} a_1 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & a_r \\ \vdots & \vdots & \ddots & \vdots \\ 0 & a_2 & \cdots & 0 \end{pmatrix}$

We can write $F = \sum_{i=1}^{r} F_i$ where $F_i$ is the matrix where an element of the $i^{th}$ column is unknown.

We follow the same approach as in the first case, we obtain

$$\dot{X}(t) = AX(t) + D\Delta EX(t) \quad (12)$$
where \( D = [D_1 \quad D_2 \quad \ldots \quad D_r] \), \( \Delta = \begin{pmatrix} \Delta_1 \\ \Delta_2 \\ \vdots \\ \Delta_r \end{pmatrix} \) and \( E = \begin{bmatrix} E_1 \\ E_2 \\ \vdots \\ E_r \end{bmatrix} \).

The condition \( \Delta' \Delta \leq I \) is satisfied if and only if \( \Delta_i' \Delta_i \leq I - a_i^2 \) for each \( i = 1, \ldots, r \).

Therefore, if \( 1 - a_i^2 \) for each \( i = 1, \ldots, r \), the system (12) is quadratically stable if and only if the LMI

\[
\overline{A}P + PA' + PDD'P + E'E < 0
\]

has a symmetric positive defined solution \( P \).

**Example 3.6** Consider the uncertain system given by

\[
A = \begin{pmatrix} -0.1 & 11 & \alpha & 1.3 \\ \beta & 2 & 5 & 0.1 \\ 1 & \gamma & 1 & 0.2 \\ 0.1 & 1 & 5 & \delta \end{pmatrix}, \quad L = [1.1 \quad 1 \quad 1]',
\]

\[
C = [4 \quad 6 \quad 5 \quad 1] \quad \text{and} \quad T = \begin{pmatrix} 2 & 0 & 5 & 3 \\ 0 & 1 & 0 & 1 \\ 1 & 4 & 3 & 0 \\ 0 & 2 & 0 & 0.1 \end{pmatrix}
\]

We use the Matlab LMI Control Toolbox, and we obtain the solution as follows

\[
P = \begin{bmatrix}
0.4945 & -2.3736 & -0.5914 & 2.3420 & 0.2572 & -0.1854 & 0.4466 & -0.5324 \\
-0.5914 & 3.0070 & 0.9686 & -3.4723 & -0.1460 & 0.2602 & -0.4364 & 0.4130 \\
0.2572 & -1.4288 & -0.1460 & 1.1043 & 0.5560 & 0.0625 & 0.1631 & -0.7669 \\
-0.1854 & -12.9141 & 0.2602 & 12.7422 & 0.0625 & 15.6722 & -1.8026 & -13.5647 \\
0.4466 & -2.0552 & -0.4364 & 1.6583 & 0.1631 & -1.8026 & 2.7456 & -1.0460 \\
-0.5324 & 16.7930 & 0.4130 & -15.6312 & -0.7669 & -13.5647 & -1.0460 & 15.6336
\end{bmatrix}
\]

**References**


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