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Extend the Borel-Cantelli Lemma to Sequences of Non-Independent Random Variables

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Abstract

In this paper we discus on the Borel-Cantelli lemma when the condition of independent events is replaced by negative quadrant dependent condition.

Keywords: Borel-cantelli Lemma, Cauchy-schwarz Inequality, Negatively Quadrant Dependent(NQD)

1. Introduction

In the probability theory, we often wish to understand the relation between events A_n in the same probability space. The first and second Borel-Cantelli Lemma and Fatou's Lemma are the important conceptions of probability theory. They have numerous applications in probability theory. We are often interested in the limits superior and limits inferior of a sequence of events An on the same probability space $(\Omega, \mathrm{A}, \mathrm{P})$. Here we prove the above lemmas when the essential condition on events (i.e. independent) is replaced by NQD. In the section two we recall some preliminaries definitions and the classical form of Borel-cantelli lemma the main result of our studies.

2. Preliminaries

Definition 1. For a sequence of subsets A_n, define the limit superior

$$\begin{aligned}\overline{\lim}\,\mathrm{A}_{\mathrm{n}} &= \limsup_{n\to\infty} A_n = \bigcap_{n=1}^{\infty}\bigcup_{k=n}^{\infty} A_k \\ &= \{\omega : \omega \in A_n \text{ for infinitly many n's}\} \\ &= \{\omega : \omega \in A_n \text{ for infinitly often}\} \\ &= \{A_n, i.o\}.\end{aligned}$$

Similarity the limit inferior is given by

$$\begin{aligned}\underline{\lim}\,\mathrm{A}_{\mathrm{n}} &= \liminf_{n\to\infty} A_n = \bigcup_{n=1}^{\infty}\bigcap_{k=n}^{\infty} A_k \\ &= \{\omega : \omega \in A_n \text{ for all but finitely many n's}\} \\ &= \{\omega : \omega \in A_n \text{ for eventually}\} \\ &= \{A_n, ev.\}.\end{aligned}$$

Remark. Note that if $A_n \in \mathrm{A}$, are measurable, then so are $\limsup_{n\to\infty} A_n$ and $\liminf_{n\to\infty} A_n$. By Demorgan,s law, we have that $\{A_n, ev.\} = \{A_n, i.o\}^c$, that is, $\omega \in A_n$ for all n large enough if and only if $\omega \in A_n$ for finitely many $n's$. Also, if $\omega \in A_n$ eventually, then certainly $\omega \in A_n$ infinitely often, that is

$$\liminf_{n\to\infty} A_n \subseteq \limsup_{n\to\infty} A_n$$

The limsup and liminf of the indicator functions on the sets satisfy in the following relations [1,3].

$$\begin{aligned}\limsup_{n\to\infty} I_{A_n}(\omega) &= I_{\limsup_{n\to\infty} A_n}(\omega) \\ \liminf_{n\to\infty} I_{A_n}(\omega) &= I_{\liminf_{n\to\infty} A_n}(\omega)\end{aligned}$$

Lemma 1. ([1]) (Borel-Cantelli Lemma I) Suppose $A_n \in \mathrm{A}$ and $\sum_{n=1}^{\infty} P(A_n) < \infty$. Then,

$$P(A_n, i.o) = 0.$$

Lemma 2. ([1]) (Borel-Cantelli Lemma II) Suppose $A_n \in \mathrm{A}$ are mutually independent And $\sum_{n=1}^{\infty} P(A_n) = \infty$. Then,

$$P(A_n, i.o) = 1.$$

Lemma 3. ([1]) (Fatou's Lemma) Let $\{A_n\}_{n=1}^{\infty}$ be events in a probability space $(\Omega, \mathrm{A}, \mathrm{P})$. Then,

$$P(\limsup_{n\to\infty} A_n) \geq \limsup_{n\to\infty} P(A_n)$$

and

$$\liminf_{n\to\infty} P(A_n) \geq P(\liminf_{n\to\infty} A_n).$$

Definition 2.([7]) A sequence of random variables $\{X_n; \mathrm{n} \geq 1\}$ is said to be pairwise negative quadrant dependent (NQD) if,

$$P(X_i \leq x, \mathrm{X_j} \leq y) \leq \mathrm{P}(X_i \leq x)\mathrm{P}(\mathrm{X_j} \leq y)$$

For all $x, \mathrm{y} \in \mathrm{R}$ and for all $\mathrm{i}, \mathrm{j} \geq 1, \mathrm{i} \neq \mathrm{j}$. For finding more details and results you can review [2], [4], [5], [6] papers. In order to prove the main theorem, we shall state the following lemma for later references. In more cases that we can studding about it that to decreases condition independence low 0-1 Borel and gain same result we extention in Theorem1, second Borel-cantelli Lemma.

3. Main result

Lemma 4. Let $Y \geq 0$ and $\mathrm{E}[\mathrm{Y}^2] < \infty$ and $E[Y] > \mathrm{a}$. Then

$$P[Y \geq \mathrm{a}] \geq \frac{(\mathrm{E}[\mathrm{Y} - \mathrm{a}])^2}{E[Y^2]}.$$

Proof. By apply the cauchy-schwarz inequality [3], one gets

$$E(Y\, \mathrm{I}_{[\mathrm{Y}>\mathrm{a}]}) \leq \{\mathrm{E}[\mathrm{Y}^2]\}^{1/2} \{E(I_{[Y>a]}^2)\}^{1/2},$$

or equivalently,

$$\{E(Y\, I_{[Y>a]})\}^2 \leq \mathrm{P}[\mathrm{Y} > \mathrm{a}]\mathrm{E}[\mathrm{Y}^2].$$

Since,

$$Y\, I_{[Y>a]} = \begin{cases} 0 & \text{if } Y \le a \\ Y & \text{if } Y > a \end{cases}.$$

It follows that,

$$Y\, I_{[Y>a]} \ge Y - a\,,$$

which, implies $E[Y\, I_{[Y>a]}] \ge E(Y) - a$, and therefore,

$$(E[Y] - a)^2 \le E[Y^2]P[Y > a].$$

Theorem 1. Let $\{A_n\}_{n=1}^{\infty}$ be a sequence of events in a probability space (Ω, A, P) and set $\limsup_{n\to\infty} A_n = A$. If $\sum_{n=1}^{\infty} P(A_n) = \infty$ and ,

$$\limsup_{n\to\infty} \frac{\left(\sum_{i=1}^{\infty} P[A_i]\right)^2}{\sum_{i=1}^{n}\sum_{j=1}^{n} P\left[A_i \bigcap A_j\right]} = \alpha > 0.$$

then,

(a) $P[A] \ge \alpha$.

(b) If the events are pairwise (NQD) then

$$P[A] = 1$$

Proof. (a) Let $Y_n = I_{A_n}$ be the indicator function of the event A_n. In addition, set $X_n = \sum_{k=1}^{n} Y_k$ and $Z_n = \frac{X_n}{E[X_n]}$. It is clear that $E[Z_n] = 1$, and,

$$E[Z_n^2] = E[Z_n Z_n] = \frac{E\left[\sum_{i=1}^{n} Y_i \sum_{j=1}^{n} Y_j\right]}{E^2[X_n]} = \frac{\sum_{i=1}^{n}\sum_{j=1}^{n} E[Y_i Y_j]}{\left(\sum_{i=1}^{n} E[Y_i]\right)^2} = \frac{\sum_{i=1}^{n}\sum_{j=1}^{n} P[A_i A_j]}{\left(\sum_{i=1}^{n} P[A_i]\right)^2}.$$

Since Z_n as defined above is a random variable that satisfies $E[Z_n] = 1$, $E[Z_n^2] < \infty$, for all $n = 1,2,\ldots$ then for any $\eta < 1$, we can use lemma 4 and write,

$$P[Z_n > \eta] \ge \frac{(E[Z_n - \eta])^2}{E[Z_n^2]} = \frac{(1-\eta)^2}{E[Z_n^2]},$$

to establish the first result is the following,

$$P(A) = P[\limsup_{n\to\infty} A_n] \ge P[\limsup Z_n > \eta].$$

To understand why this is so essentially, we prove that

$$A^c \subseteq \{\omega : \limsup_{n\to\infty} Z_n(\omega) > \eta\}^c.$$

This gives,

$$A \supseteq \{\omega : \limsup_{n\to\infty} Z_n(\omega) > \eta\}.$$

Assume that $\omega \notin A$. This means that $\lim_{n\to\infty} X_n(\omega)$, is a finite number end, thus, $\lim_{n\to\infty} Z_n(\omega) = 0$, since by assumption,

$$E[X_n] = \sum_{i=1}^{n} P[A_i] \to \infty \quad \text{as, n} \to \infty .$$

This is the same as saying that for any $\eta > 0$, $Z_n(\omega) \le \eta$ i.o. Hence, if $\omega \in A$, it must satisfy the requirement $Z_n(\omega) > \eta$ i.o. or, say

$$\omega \in \{\omega : \limsup_{n\to\infty} Z_n(\omega) > \eta\}$$

Now using by Lemma 4, we can write,

$$\begin{aligned} P[\limsup_{n\to\infty} Z_n > \eta] &\ge \limsup_{n\to\infty} P[Z > \eta] \\ &\ge \limsup_{n\to\infty} \frac{1-\eta^2}{E[Z_n^2]} \\ &= (1-\eta^2)\alpha . \end{aligned}$$

Using the second assumption of the theorem. Since . Is an arbitrary number in (0, 1), it follows that $P[A] \ge \alpha$ as we were supposed to show.

(b) If we introduce the extra assumption that the events $\{A_n\}_{n\ge1}^{\infty}$ are pairewise NQD, then

$$\begin{aligned} \frac{\left(\sum_{i=1}^{n} P[A_i]\right)^2}{\sum_{i=1}^{n}\sum_{j=1}^{n} P[A_j \bigcap A_j]} &\ge \frac{\sum_{i=1}^{n} P^2[A_i] + \sum_{i=1}^{n}\sum_{j=1, i\ne j}^{n} P[A_i]P[A_j]}{\sum_{i=1}^{n} P[A_i] + \sum_{i=1}^{n}\sum_{j=1, i\ne j}^{n} P[A_i]P[A_j]} \\ &\ge \frac{\sum_{i=1}^{n}\sum_{j=1, i\ne j}^{n} P[A_i]P[A_j]}{\sum_{i=1}^{n} P[A_i] + \sum_{i=1}^{n}\sum_{j=1, i\ne j}^{n} P[A_i]P[A_j]} \\ &= \frac{1}{1 + \dfrac{\sum_{i=1}^{n} P[A_i]}{\left(\sum_{i=1}^{n} P[A_i]\right)^2 - \sum_{i=1}^{n} P^2[A_i]}} . \end{aligned}$$

Now since,

$$\frac{\sum_{i=1}^{n} P[A_i]}{\left(\sum_{i=1}^{n} P[A_i]\right)^2 - \sum_{i=1}^{n} P^2[A_i]} = \frac{1}{\sum_{i=1}^{n} P[A_i] - \frac{\sum_{i=1}^{n} P^2[A_i]}{\sum_{i=1}^{n} P[A_i]}} \to 0 \quad \text{as}, \text{n} \to \infty.$$

By assumption, $\sum_{n=1}^{\infty} P(A_n) = \infty$ we can use the statement just prove to write

$$P[A] \geq \limsup_{n\to\infty} \frac{\left(\sum_{i=1}^{n} P[A_i]\right)^2}{\sum_{i=1}^{n}\sum_{j=1}^{n} P[A_j \bigcap A_j]} = 1 \ .$$

Hence $P[A] = 1$.

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