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Existence of Multiple Positive Solutions for Even Order Sturm-Liouville Dynamic Equations

A. Kameswara Rao

Department of Applied Mathematics
Andhra University, Visakhapatnam, 530003, India
kamesh_1724@yahoo.com

S. Nageswara Rao

Department of Mathematics
Sri Prakash College of Engineering
Tuni, 533 401, A. P, India
sabbavarapu_nag@yahoo.co.in

Abstract

We consider the even-order dynamic equation on time scales

$$(-1)^n y^{(\Delta\nabla)^n}(t) = f(t, y(t)), \quad t \in [a, b]$$

satisfying the boundary conditions

$$\alpha_{i+1} y^{(\Delta\nabla)^i}(a) - \beta_{i+1} y^{(\Delta\nabla)^i \Delta}(a) = 0, \quad \gamma_{i+1} y^{(\Delta\nabla)^i}(b) + \delta_{i+1} y^{(\Delta\nabla)^i \Delta}(b) = 0$$

for $0 \le i \le n-1$, $f : [a, b] \times \mathbb{R} \to \mathbb{R}$ is continuous. First, we establish the existence of at least three positive solutions by using the well-known Leggett-Williams fixed point theorem. We also establish the existence of at least 2m-1 positive solutions for arbitrary positive integer m.

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Keywords: Dynamic equation, green's function, positive solution, multiple positive solution, cone

1 Introduction

In this paper we consider the existence of positive solutions for even-order dynamic equation on time sclaes

$$(-1)^n y^{(\Delta\nabla)^n}(t) = f(t, y(t)), \quad t \in [a, b] \tag{1.1}$$

satisfying Sturm-Liouville like boundary conditions

$$\alpha_{i+1}y^{(\Delta\nabla)^i}(a)-\beta_{i+1}y^{(\Delta\nabla)^i\Delta}(a)=0,\quad \gamma_{i+1}y^{(\Delta\nabla)^i}(b)+\delta_{i+1}y^{(\Delta\nabla)^i\Delta}(b)=0. \tag{1.2}$$

Here $n \geq 1$, $0 \leq i \leq n-1$, with $a \in \mathbb{T}_{k^n}$, $b \in \mathbb{T}^{k^n}$ for a time scale $\mathbb{T}$ and $\sigma^n(a) < \rho^n(b)$. We take
(A) α_j, β_j, γ_j, $\delta_j \geq 0$ and $d_j = \gamma_j\beta_j + \alpha_j\delta_j + \alpha_j\gamma_j(b-a) > 0$.

The study of the existence of positive solutions of the even order boundary value problems (BVPs) arises in a variety of different areas of applied mathematics and physics. In the modeling of nonlinear diffusion via nonlinear sources, thermal ignition of gases, and in chemical concentrates in biological problem [17]. In these applied settings, only positive solutions are meaningful. The existence of positive solutions are studied by many authors. To mention a few, we list some papers, Eloe and Henderson [13, 14, 15], Erbe and Wang [17] for at least one positive solution and then Anderson and Avery [5], Avery and Peterson [10], Henderson and Kaufmann [18] for multiple positive solutions.

This paper is organized as follows. In Section 2, we state some preliminaries on time scales. In Section 3, we state and prove some lemmas which are needed in our main results. In Section 4, we establish the existence of at least three positive solutions of the BVP (1.1)-(1.2) by using Leggett-Williams fixed point theorem. In Section 5, we establish the existence of at least $2m-1$ positive solutions of the BVP (1.1)-(1.2) for arbitrary positive integer m.

2 Preliminaries about Time Scales

A time scale $\mathbb{T}$ is a nonempty closed subset of $\mathbb{R}$. Hilger [19] initially introduced time scales with the twin goals of unifying the continuous and discrete calculus and extending the results to a dynamic calculus for general time scales. For excellent introduction to the overall area of dynamic equations on time scales, see the recent texts by Bohner and Peterson [12], from which we cull the following definitions. The functions $\sigma, \rho : \mathbb{T} \to \mathbb{T}$ are jump operators given by

$$\sigma(t) = \inf\{s \in \mathbb{T} : s > t\} \ \text{ and } \ \rho(t) = \sup\{s \in \mathbb{T} : s < t\}$$

(supplemented by $\inf\emptyset = \sup\mathbb{T}$ and $\sup\emptyset = \inf\mathbb{T}$). The point $t \in \mathbb{T}$ is left-dense, left-scattered, right-dense, right-scattered if $\rho(t) = t$, $\rho(t) < t$, $\sigma(t) = t$, $\sigma(t) > t$, respectively. If $\mathbb{T}$ has a right-scattered minimum m, define $\mathbb{T}_k = \mathbb{T} - \{m\}$; otherwise, set $\mathbb{T}_k = \mathbb{T}$. If $\mathbb{T}$ has a left-scattered maximum M, define $\mathbb{T}^k = \mathbb{T} - \{M\}$; otherwise, set $\mathbb{T}^k = \mathbb{T}$.

For $f : \mathbb{T} \to \mathbb{R}$ and $t \in \mathbb{T}^k$, the delta derivative of f at t, denoted $f^\Delta(t)$, is the number (provided it exists) with the property the given any $\epsilon > 0$, there is a neighborhood U of t such that

$$| f(\sigma(t)) - f(s) - f^{\Delta}(t)[\sigma(t) - s] | \leq \epsilon | \sigma(t) - s |$$

for all $s \in U$.

For $f : \mathbb{T} \to \mathbb{R}$ and $t \in \mathbb{T}_k$, the nabla derivative of f at t, denoted $f^{\nabla}(t)$, is the number (provided it exists) with the property the given any $\epsilon > 0$, there is a neighborhood U of t such that

$$| f(\rho(t)) - f(s) - f^{\nabla}(t)[\rho(t) - s] | \leq \epsilon | \rho(t) - s |$$

for all $s \in U$.

A function $f : \mathbb{T} \to \mathbb{R}$ is left-dense continuous or ld-continuous on $[a, b]$, denoted $f \in C_{ld}[a, b]$, provided it is continuous at left-dense points in $\mathbb{T}$ and its right-sided limits exist (finite) at right-dense points in $\mathbb{T}$. It is known that if f is ld-continuous, then there is a function $F(t)$ such that $F^{\nabla}(t) = f(t)$. In this case, we define

$$\int_a^b f(t)\nabla t = F(b) - F(a).$$

3 The Green's function and Bounds

To state and prove the main results of this paper, we need the following lemmas. Let $G_j(t, s)$ be the Green's function for the boundary value problems,

$$-y^{\Delta\nabla}(t) = 0, \quad t \in [a, b] \tag{3.1}$$

$$\alpha_j y(a) - \beta_j y^{\Delta}(a) = 0, \quad \gamma_j y(b) + \delta_j y^{\Delta}(b) = 0. \tag{3.2}$$

for $1 \leq j \leq n$. First, we need few results on the related second order homogeneous boundary value problem (3.1)-(3.2).

Lemma 3.1 *For $1 \leq j \leq n$, let*

$$d_j = \gamma_j\beta_j + \alpha_j\delta_j + \alpha_j\gamma_j(b - a).$$

The homogeneous boundary value problem (3.1)-(3.2) *has only the trivial solution if and only if $d_j > 0$.*

Lemma 3.2 *For $1 \leq j \leq n$, the Green's function $G_j(t, s)$ for the homogeneous boundary value problem* (3.1)-(3.2), *is given by*

$$G_j(t, s) = \begin{cases} \frac{1}{d_j}\{\alpha_j(t - a) + \beta_j\}\{\gamma_j(b - s) + \delta_j\} : & \mathrm{a \leq t \leq s \leq b} \\ \\ \frac{1}{d_j}\{\alpha_j(s - a) + \beta_j\}\{\gamma_j(b - t) + \delta_j\} : & \mathrm{a \leq s \leq t \leq b} \end{cases} \tag{3.3}$$

Lemma 3.3 *Assume that condition (A) is satisfied. Then, the Green's function $G_j(t,s)$ satisfies the following inequality*

$$g_j(t)G_j(s,s) \leq G_j(t,s) \leq G_j(s,s), \quad for\ any \quad s,t \in [a,b], \tag{3.4}$$

where

$$g_j(t) = min\left\{\frac{\alpha_j(t-a)+\beta_j}{\alpha_j(b-a)+\beta_j}, \quad \frac{\gamma_j(b-t)+\delta_j}{\gamma_j(b-a)+\delta_j}\right\} < 1, \tag{3.5}$$

for $1 \leq j \leq n$.

Proof: It is straightforward to tee that

$$\frac{G_j(t,s)}{G_j(s,s)} = \begin{cases} \frac{\alpha_j(t-a)+\beta_j}{\alpha_j(s-a)+\beta_j} : & \text{a} \leq \text{t} \leq \text{s} \leq \text{b} \\ \\ \frac{\gamma_j(b-t)+\delta_j}{\gamma_j(b-s)+\delta_j} : & \text{a} \leq \text{s} \leq \text{t} \leq \text{b} \end{cases}$$

this expression yields both inequalities in (3.4) for g_j as in (3.5). □

Lemma 3.4 *Assume that the condition (A) is satisfied, and $G_j(t,s)$ as in (3.4). Let us define $H_1(t,s) = G_1(t,s)$, and recursively define*

$$H_j(t,s) = \int_a^b H_{j-1}(t,r)G_j(r,s)\nabla r, \tag{3.6}$$

for $2 \leq j \leq n$. Then $H_n(t,s)$ is the Green's function for the corresponding homogeneous problem (1.1)-(1.2).

Let ξ and ω are chosen from $\mathbb{T}$ such that $a < \xi < \omega < b$ and also

$$m_j = \min_{t\in[\xi,\omega]} g_j(t) \tag{3.7}$$

for g_j as in (3.5).

Lemma 3.5 *Assume that the condition (A) holds. If we define*

$$K = \prod_{j=1}^{n-1} K_j, \quad L = \prod_{j=1}^{n-1} m_j L_j,$$

then the Green's function $H_n(t,s)$ in Lemma 3.4 *satisfies*

$$0 \leq H_n(t,s) \leq KG_n(s,s), \quad (t,s) \in [a,b]\times[a,b]$$

and

$$H_n(t,s) \geq m_n L G_n(s,s), \quad (t,s) \in [\xi,\omega]\times[a,b]$$

where m_n *is given in* (3.7),

$$K_j = \int_a^b G_j(s,s)\nabla s > 0, \quad 1 \le j \le n,$$

and

$$L_j = \int_\xi^\omega G_j(s,s)\nabla s > 0, \quad 1 \le j \le n.$$

Proof: We using mathematical induction on n it is straightforward. □

4 Existence of at least three positive solutions

In this section, we establish the existence of at least three positive solutions for two point even order dynamic equations on time scales (1.1)-(1.2), by using Leggett-Williams fixed point theorem.

Let E be a real Banach space with cone P. A map $S : P \to [0,\infty)$ is said to be a nonnegative continuous concave functional on P, if S is continuous and

$$S(\lambda x + (1-\lambda)y) \ge \lambda S(x) + (1-\lambda)S(y)$$

for all $x, y \in P$ and $\lambda \in [0,1]$. Let α and β be two numbers such that $0 < \alpha < \beta$ and S be a nonnegative continuous concave functional on P. We define the following convex sets

$$P_\alpha = \{y \in P : \| y \| < \alpha\}$$

$$P(S,\alpha,\beta) = \{y \in P : \alpha \le S(y), \| y \| \le \beta\}$$

Theorem 4.1 [Leggett-Williams fixed point theorem] *Let* $T : \overline{P}_{a_3} \to \overline{P}_{a_3}$ *be completely continuous and* S *be a nonnegative continuous concave functional on* P *such that* $S(y) \le \| y \|$ *for all* $y \in \overline{P}_{a_3}$*. Suppose that there exist* $0 < d < a_1 < a_2 \le a_3$ *such that*
(i)$\{y \in P(S,a_1,a_2) : S(y) > a_1\} \ne \emptyset$ *and* $S(Ty) > a_1$ *for* $y \in P(S,a_1,a_2)$;
(ii) $\| Ty \| < d$ *for* $\| y \| \le d$;
(iii)$S(Ty) > a_1$ *for* $y \in P(S,a_1,a_3)$ *with* $\| Ty \| > a_2$.
Then T *has at least three fixed points* y_1, y_2, y_3 *in* $\overline{P}_{a_3}$ *satisfying*

$$\| y_1 \| < d, a_1 < S(y_2), \| y_3 \| > d, S(y_3) < a_1.$$

$$\text{Let} \qquad M = m_n \prod_{j=1}^{n-1} \frac{m_j L_j}{K_j}.$$

Theorem 4.2 *Assume that there exist numbers* a_0, a_1, *and* a_2 *with* $0 < a_0 < a_1 < \frac{a_1}{M} < a_2$ *such that*

$$f(t, y(t)) < \frac{a_0}{\prod_{j=1}^{n} K_j}, \ for\ t \in [a, b]\ and\ y \in [0, a_0], \tag{4.1}$$

$$f(t, y(t)) > \frac{a_1}{m_n \prod_{j=1}^{n} L_j}, \ for\ t \in [\xi, \omega]\ and\ y \in [a_1, \frac{a_1}{M}], \tag{4.2}$$

$$f(t, y(t)) < \frac{a_2}{\prod_{j=1}^{n} K_j}, \ for\ t \in [a, b]\ and\ y \in [0, a_2]. \tag{4.3}$$

Then the BVP (1.1)-(1.2) *has at least three positive solutions.*

Proof: Let the Banach Space $E = C[a, b]$ be equipped with the norm

$$\| y \| = \max_{t \in [\xi, \omega]} | y(t) | .$$

We denote

$$P = \{y \in E : y(t) \geq 0,\ t \in [a, b]\}.$$

Then, it is obvious that P is a cone in E. For $y \in P$, we define

$$S(y) = \min_{t \in [\xi, \omega]} | y(t) |, \quad \text{and}$$

$$Ty(t) = \int_a^b H_n(t, s) f(s, y(s)) \nabla s, \quad t \in [a, b].$$

It is easy to check that S is a nonnegative continuous concave functional on P with $S(y) \leq \| y \|$ for $y \in P$ and that $T : P \to P$ is completely continuous and fixed points of T are solutions of the BVP (1.1)-(1.2). First, we prove that, if there exists a positive number r such that $f(t, y(t)) < \frac{r}{\prod_{j=1}^{n} K_j}$ for $t \in [a, b]$ and $y \in [0, r]$, then $T : \overline{P}_r \to P_r$. Indeed, if $y \in \overline{P}_r$, then for $t \in [a, b]$ we have

$$\begin{aligned} Ty(t) &= \int_a^b H_n(t, s) f(s, y(s)) \nabla s \\ &< \frac{r}{\prod_{j=1}^{n} K_j} \int_a^b H_n(t, s) \nabla s \\ &\leq \frac{r}{\prod_{j=1}^{n} K_j} K \int_a^b G_n(s, s) \nabla s = r. \end{aligned}$$

Thus, $\| Ty \| < r$, that is, $Ty \in P_r$. Hence, we have shown that if (4.1) and (4.3) hold, then T maps $\overline{P}_{a_0}$ into P_{a_0} and $\overline{P}_{a_2}$ into P_{a_2}. Next, we show that

$\{y \in P(S, a_1, \frac{a_1}{M}) : S(y) > a_1\} \neq \emptyset$ and $S(Ty) > a_1$ for all $y \in P(S, a_1, \frac{a_1}{M})$. In fact, the constant function

$$\frac{a_1 + a_1/M}{2} \in \{y \in P(S, a_1, \frac{a_1}{M}) : S(y) > a_1\}.$$

Moreover, for $y \in P(S, a_1, \frac{a_1}{M})$, we have

$$\frac{a_1}{M} \geq \| y \| \geq y(t) \geq \min_{t \in [\xi, \omega]} y(t) = S(y) \geq a_1$$

for all $t \in [\xi, \omega]$. Thus, in view of (4.2) we see that

$$\begin{aligned} S(Ty) &= \min_{t \in [\xi, \omega]} \int_a^b H_n(t,s) f(s, y(s)) \nabla s \\ &\geq \min_{t \in [\xi, \omega]} \int_\xi^\omega H_n(t,s) f(s, y(s)) \nabla s \\ &> \frac{a_1}{m_n \prod_{j=1}^n L_j} m_n L \int_\xi^\omega G_n(s,s) \nabla s = a_1 \end{aligned}$$

as required. Finally, we show that if $y \in P(S, a_1, a_2)$ and $\| Ty \| > \frac{a_1}{M}$, then $S(Ty) > a_1$. To see this, we suppose that $y \in P(S, a_1, a_2)$ and $\| Ty \| > \frac{a_1}{M}$, then, by Lemma 3.5, we have

$$\begin{aligned} S(Ty) &= \min_{t \in [\xi, \omega]} \int_a^b H_n(t,s) f(s, y(s)) \nabla s \\ &\geq \min_{t \in [\xi, \omega]} m_n L \int_a^b G_n(s,s) f(s, y(s)) \nabla s \\ &\geq m_n L \int_\xi^\omega G_n(s,s) f(s, y(s)) \nabla s \end{aligned}$$

for all $t \in [a, b]$. Thus

$$S(Ty) \geq \frac{m_n L}{K} \max_{t \in [a,b]} \int_a^b H_n(t,s) f(s, y(s)) \nabla s = \frac{m_n L}{K} \| Ty \| > \frac{m_n L}{K} \frac{a_1}{M} = a_1.$$

To sum up, all the hypotheses of Theorem 4.1 are satisfied. Hence T has at least three fixed points, that is, the BVP(1.1)-(1.2) has at least three positive solutions y_1, y_2 and y_3 such that

$$\| y_1 \| < a_0, a_1 < \min_{t \in [\xi, \omega]} y_2(t), \| y_3 \| > a_0, \min_{t \in [\xi, \omega]} y_3(t) < a_1.$$

□

5 Existence of multiple positive solutions

In this section, we establish the existence of multiple positive solutions for the BVP (1.1)-(1.2), by using induction on m.

Theorem 5.1 *Let m be an arbitrary positive integer. Assume that there exist numbers $a_i (1 \leq i \leq m)$ and $b_j (1 \leq j \leq m-1)$ with $0 < a_1 < b_1 < \frac{b_1}{M} < a_2 < b_2 < \frac{b_2}{M} < ... < a_{m-1} < b_{m-1} < \frac{b_{m-1}}{M} < a_m$ such that*

$$f(t, y(t)) < \frac{a_i}{\prod_{j=1}^{n} K_j}, \ \textit{for } t \in [a, b] \textit{ and } y \in [0, a_i], 1 \leq i \leq m \tag{5.1}$$

$$f(t, y(t)) > \frac{b_j}{m_n \prod_{j=1}^{n} L_j}, \ \textit{for } t \in [\xi, \omega] \textit{ and } y \in [b_j, \frac{b_j}{M}], 1 \leq j \leq m-1 \tag{5.2}$$

Then the BVP (1.1)-(1.2) has at least $2m-1$ positive solutions in $\overline{P}_{a_m}$.

Proof: We use induction on m. First, for $m = 1$, we know from (5.1) that $T : \overline{P}_{a_1} \to P_{a_1}$, then, it follows from Schauder fixed point theorem that the BVP (1.1)-(1.2) has at least one positive solution in $\overline{P}_{a_1}$. Next, we assume that this conclusion holds for $m = k$. In order to prove that this conclusion holds for $m = k + 1$, we suppose that there exist numbers $a_i (1 \leq i \leq k+1)$ and $b_j (1 \leq j \leq k)$ with $0 < a_1 < b_1 < \frac{b_1}{M} < a_2 < b_2 < \frac{b_2}{M} < ... < a_k < b_k < \frac{b_k}{M} < a_{k+1}$ such that

$$f(t, y(t)) < \frac{a_i}{\prod_{j=1}^{n} K_j}, \ \text{for} \ \ t \in [a, b] \ \text{and} \ \ y \in [0, a_i], 1 \leq i \leq k+1, \tag{5.3}$$

$$f(t, y(t)) > \frac{b_j}{m_n \prod_{j=1}^{n} L_j}, \ \text{for} \ \ t \in [\xi, \omega] \ \text{and} \ \ y \in [b_j, \frac{b_j}{M}], 1 \leq j \leq k \tag{5.4}$$

By assumption, the BVP (1.1)-(1.2) has at least $2k-1$ positive solutions $u_i (i = 1, 2, ..., 2k-1)$ in $\overline{P}_{a_k}$. At the same time, it follows from Theorem 4.2, (5.3) and (5.4) that the BVP (1.1)-(1.2) has at least three positive solutions u, v and w in $\overline{P}_{a_{k+1}}$ such that, $\| u \| < a_k, b_k < \min_{t \in [\xi, \omega]} v(t), \| w \| > a_k, \min_{t \in [\xi, \omega]} w(t) < b_k$. Obviously, v and w are different from $u_i (i = 1, 2, ..., 2k-1)$. Therefore, the BVP(1.1)-(1.2) has at least $2k+1$ positive solutions in $\overline{P}_{a_{k+1}}$ which shows that this conclusion also holds for $m = k + 1$. □

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