Existence and Approximation of Solutions for Nonlocal Boundary Value Problems with Mixed Nonlinearities

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Abstract

In this paper, we study a second-order boundary value problem involving mixed nonlinearities in the differential equation and nonlinear nonlocal general three-point boundary conditions. We develop a generalized quasilinearization technique to obtain monotone sequences of lower and upper solutions converging quadratically to a unique solution of the problem at hand.

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1 Introduction

The subject of multi-point nonlocal boundary value problems, initiated by Ilin and Moiseev [1,2], has been addressed by many authors, for instance, [3-9]. In particular, Eloe and Gao [10] discussed the quasilinearization method for a three-point nonlinear boundary value problem. The quasilinearization technique [11] is quite fruitful as it not only proves the existence of the solutions of the problem but also provides an iterative scheme with quadratic convergence. The nineties brought new dimensions to this technique when Lakshmikantham

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generalized the method of quasilinearization by relaxing the convexity assumption, see [12] and the references therein. This method has been developed for a variety of initial and boundary value problems, see [13-26].

In this paper, we consider a nonlinear second order ordinary differential equation with nonlinear nonlocal boundary conditions

\[
x''(t) = N(t, x(t)), \quad t \in J = [0, 1],
\]

\[
\alpha x(0) - \beta x'(0) = g_1(x(\sigma)), \quad \alpha x(1) + \beta x'(1) = g_2(x(\sigma)), \quad 0 < \sigma < 1,
\]

where \(N: J \times \mathbb{R} \to \mathbb{R}\) is continuous, \(g_i: \mathbb{R} \to \mathbb{R}\) \((i = 1, 2)\) are continuous and \(\alpha, \beta > 0\) with \(\alpha > 1\). The nonlinearity \(N(t, x)\) in (1.1) is assumed to be of the form

\[
N(t, x) = f(t, x) + k(t, x) + H(t, x),
\]

where \(f(t, x)\) is not convex but \(f(t, x) + \phi(t, x)\) is convex for some convex function \(\phi(t, x)\), \(k(t, x)\) is not concave but \(k(t, x) + \chi(t, x)\) is concave for some concave function \(\chi(t, x)\), and \(H(t, x)\) is only Lipschitzian.

The importance of the work lies in the fact that the nonlinear part of the differential equation consists of mixed type of nonlinearities (which do not necessarily satisfy the convexity/concavity assumptions) and a Lipschitz function. The boundary conditions of the type (1.2) appear in certain problems of thermodynamics and wave propagation where the controllers at the boundary points \(t = 0\) and \(t = 1\) dissipate or add energy according to a censor located at any interior position \(t = \sigma\) \((0 < \sigma < 1)\). The method of generalized quasilinearization is applied to obtain sequences of approximate solutions converging monotonically and quadratically to the unique solution of (1.1) and (1.2).

## 2 Preliminary results

**Definition 2.1.** A function \(u \in C^2[J, \mathbb{R}]\) is a lower solution of the BVP (1.1)-(1.2) if

\[
u''(t) \geq N(t, u), \quad t \in J,
\]

\[
\alpha u(0) - \beta u'(0) \leq g_1(u(\sigma)), \quad \alpha u(1) + \beta u'(1) \leq g_2(u(\sigma)),
\]

and \(v \in C^2[J, \mathbb{R}]\) is an upper solution of the BVP (1.1)-(1.2) if

\[
u''(t) \leq N(t, v), \quad t \in J,
\]

\[
\alpha v(0) - \beta v'(0) \geq g_1(v(\sigma)), \quad \alpha v(1) + \beta v'(1) \geq g_2(v(\sigma)).
\]
Existence and approximation of solutions

By the Green’s function method, the solution \( u(t) \) of the boundary value problem (1.1)-(1.2), can be written as

\[
x(t) = \left( \frac{1}{\alpha^2 + 2\alpha \beta} \right) \left[ g_1(x(\sigma)) \left( \beta + \alpha(1-t) \right) + g_2(x(\sigma)) \left( \beta + \alpha t \right) \right] + \int_0^1 G(t, s) N(s, x(s)) ds,
\]

where

\[
G(t, s) = \frac{1}{\alpha^2 + 2\alpha \beta} \begin{cases} 
(\alpha t + \beta) (\alpha(s - 1) - \beta), & \text{if } 0 \leq t \leq s \leq 1, \\
(\alpha(t - 1) - \beta) (\alpha s + \beta), & \text{if } 0 \leq s \leq t \leq 1.
\end{cases}
\]

We note that \( G(t, s) < 0 \) on \( J \times J \).

The following results are necessary to prove the main result. We do not provide the proof of these results as it is based on a standard procedure, see for instance [10, 12].

**Theorem 2.1.** (Comparison result)
Assume that \( N \) is continuous with \( N_x > 0 \) on \( J \times \mathbb{R} \) and \( g_i \) are continuous on \( \mathbb{R} \) satisfying one-sided Lipschitz condition:

\[
g_i(x) - g_i(y) \leq L_i(x - y), \quad 0 \leq L_i < 1, \quad i = 1, 2.
\]

Let \( u \) and \( v \) be respectively lower and upper solutions of (1.1)-(1.2). Then \( u(t) \leq v(t), \ t \in J \).

**Theorem 2.2.** (Existence result)
Assume that \( N \) is continuous on \( J \times \mathbb{R} \) and \( g_i \) are continuous on \( \mathbb{R} \) satisfying one-sided Lipschitz condition:

\[
g_i(x) - g_i(y) \leq L_i(x - y), \quad 0 \leq L_i < 1, \quad i = 1, 2.
\]

Further, there exist lower and upper solution \( u \) and \( v \) respectively of (1.1)-(1.2) such that \( u(t) \leq v(t), \ t \in J \). Then, there exists a solution \( x(t) \) of (1.1)-(1.2) such that \( u(t) \leq x(t) \leq v(t), \ t \in J \).

### 3 Main result

**Theorem 3.1** Assume that

\((A_1)\) \( u_0, v_0 \in C^2[J, \mathbb{R}] \) are lower and upper solutions of (1.1)-(1.2) respectively.
(A₂) $N \in C[J \times \mathbb{R}, \mathbb{R}]$ be such that

$$N(t, x) = f(t, x) + k(t, x) + H(t, x),$$

where $f_x(t, x), k_x(t, x), f_{xx}(t, x), k_{xx}(t, x)$ exist and are continuous with $(f_{xx}(t, x) + \phi_{xx}(t, x)) \geq 0$, $(k_{xx}(t, x) + \chi_{xx}(t, x)) \leq 0$ for every $(t, x) \in S = \{(t, x) \in J \times \mathbb{R} : u_0(t) \leq x(t) \leq v_0(t)\}$ and $\phi, \chi \in C[J \times \mathbb{R}, \mathbb{R}]$ are such that $\phi_{xx} \geq 0$, $\chi_{xx} \leq 0$ on $S$. Further, $H(t, x)$ satisfies one sided Lipschitz condition:

$$H(t, x) - H(t, y) \geq -L(x - y), \quad x \geq y, \quad x, y \in \mathbb{R},$$

where $L > 0$ is a Lipschitz constant. Moreover, $f_x(t, x) + k_x(t, x) - L \geq 0$ for every $(t, x) \in S$.

(A₃) $g_i, g'_i, g''_i$ are continuous on $\mathbb{R}$ satisfying $0 \leq g'_i \leq 1$ and $(g''_i(x) + \psi''_i(x)) \leq 0$ with $\psi''_i \leq 0$ ($i = 1, 2$) on $\mathbb{R}$ for some continuous functions $\psi_i(x)$.

Then there exist monotone sequences $\{u_n\}$ and $\{v_n\}$ that converge in the space of continuous functions on $J$ quadratically to a unique solution $x(t)$ of the BVP (1.1)-(1.2).

Proof. Let us define $F : J \times \mathbb{R} \to \mathbb{R}$ by $F(t, x) = f(t, x) + \phi(t, x)$, $K : J \times \mathbb{R} \to \mathbb{R}$ by $K(t, x) = k(t, x) + \chi(t, x)$, $G_i : \mathbb{R} \to \mathbb{R}$ by $G_i(x) = g_i(x) + \psi_i(x), \quad i = 1, 2$. Clearly $F_{xx}(t, x) \geq 0$, $K_{xx}(t, x) \leq 0$, and $G''_i(x) \leq 0, \quad i = 1, 2$. Using the mean value theorem together with (A₂) and (A₃), we obtain

$$f(t, x) \geq f(t, y) + F_x(t, y)(x - y) - \phi(t, x) + \phi(t, y), \quad (3.1)$$
$$k(t, x) \leq k(t, y) + K_x(t, y)(x - y) - \chi(t, x) + \chi(t, y), \quad (3.2)$$
$$g_i(x) \leq g_i(y) + G'_i(y)(x - y) + \psi_i(y) - \psi_i(x), \quad i = 1, 2. \quad (3.3)$$

From (3.1) and (3.2), it follows that

$$f(t, y) + k(t, y) + F_x(t, y)(x - y) + K_x(t, x)(x - y) + \phi(t, x) - \phi(t, y) + \phi(t, y) - \phi(t, x) + \chi(t, x) - \phi(t, x)$$
$$- \chi(t, x) + H(t, x) \leq f(t, x) + k(t, x) + H(t, x) \leq f(t, y) + k(t, y) + F_x(t, x)(x - y)$$
$$+ K_x(t, y)(x - y) + \phi(t, x) - \phi(t, x) + \chi(t, y) - \chi(t, x) + H(t, x). \quad (3.4)$$

We set

$$A(t, x; u_0, v_0) = f(t, u_0) + k(t, u_0) + H(t, x)$$
$$+ [F_x(t, v_0) + K_x(t, u_0) - \phi_x(t, u_0) - \chi_x(t, v_0)](x - u_0), \quad (3.5)$$
Using (3.3), we have

\[ B(t, x; u_0, v_0) = f(t, v_0) + k(t, v_0) + H(t, x) + \left[ F_x(t, v_0) + K_x(t, u_0) - \phi_x(t, u_0) - \chi_x(t, v_0) \right](x - v_0), \tag{3.6} \]

and for \( i = 1, 2, \)

\[
\begin{align*}
  h_i(x(\sigma); u_0, v_0) &= g_i(u_0(\sigma)) + G_i(v_0(\sigma))(x(\sigma) - u_0(\sigma)) + \psi_i(u_0(\sigma)) - \psi_i(x(\sigma)), \\
  \hat{h}_i(x(\sigma); v_0) &= g_i(v_0(\sigma)) + G_i(v_0(\sigma))(x(\sigma) - v_0(\sigma)) + \psi_i(v_0(\sigma)) - \psi_i(x(\sigma)).
\end{align*}
\]

Observe that

\[
A(t, u_0; u_0, v_0) = N(t, u_0), \quad N(t, x) \leq A(t, x; u_0, v_0), \tag{3.7}
\]

\[
h_i(u_0(\sigma); u_0, v_0) = g_i(u_0(\sigma)), \quad g_i(x) \geq h_i(x(\sigma); u_0, v_0), \quad i = 1, 2, \tag{3.8}
\]

and

\[
B(t, v_0; u_0, v_0) = N(t, v_0), \quad N(t, x) \geq B(t, x; u_0, v_0), \tag{3.9}
\]

\[
\hat{h}_i(v_0(\sigma); v_0) = g_i(v_0(\sigma)), \quad g_i(x) \leq \hat{h}_i(x(\sigma); v_0), \quad i = 1, 2. \tag{3.10}
\]

In view of the assumption \((A_3)\) and (3.3), we obtain

\[ 0 \leq g_i'(x) \leq G_i'(y) - \psi_i'(x) \leq \hat{h}_i' \leq G_i'(v_0(\sigma)) - \psi_i'(v_0(\sigma)) = g_i'(v_0(\sigma)) \leq 1. \]

Similarly, it can be shown that \( 0 \leq \hat{h}_i' \leq 1. \)

Now, we consider the BVP

\[
x''(t) = A(t, x; u_0, v_0), \quad t \in J, \tag{3.11}
\]

\[
\alpha x(0) - \beta x'(0) = h_1(x(\sigma); u_0, v_0), \quad \alpha x(1) + \beta x'(1) = h_2(x(\sigma); u_0, v_0). \tag{3.12}
\]

Using \((A_1), (3.7)\) and \((3.8),\) we have

\[
\begin{align*}
  v_0''(t) &\geq N(t, u_0(t)) = A(t, u_0; u_0, v_0), \\
  \alpha v_0(0) - \beta v_0'(0) &\leq g_1(u_0(\sigma)) = h_1(u_0(\sigma); u_0, v_0), \\
  \alpha v_0(1) + \beta v_0'(1) &\leq g_2(u_0(\sigma)) = h_2(u_0(\sigma); u_0, v_0),
\end{align*}
\]

and

\[
\begin{align*}
  v_0''(t) &\leq N(t, v_0(t)) \leq A(t, v_0; u_0, v_0), \\
  \alpha v_0(0) - \beta v_0'(0) &\geq g_1(v_0(\sigma)) = h_1(v_0(\sigma); u_0, v_0), \\
  \alpha v_0(1) + \beta v_0'(1) &\geq g_2(v_0(\sigma)) = h_2(v_0(\sigma); u_0, v_0),
\end{align*}
\]
which imply that \( u_0 \) is a lower solution and \( v_0 \) is an upper solution of \((3.11)-(3.12)\). Hence, by Theorems 2.1 and 2.2, there exists a unique solution \( u_1 \) of \((3.11)-(3.12)\) such that
\[
 u_0(t) \leq u_1(t) \leq v_0(t), \quad t \in J. 
\] (3.13)
Next, consider the BVP
\[
x''(t) = B(t,x;u_0,v_0), \quad t \in J, 
\] (3.14)
\[
\alpha x(0) - \beta x'(0) = \hat{h}_1(x(\sigma);v_0), \quad \alpha x(1) + \beta x'(1) = \hat{h}_2(x(\sigma);v_0). 
\] (3.15)
Using \((A_1), (3.9)\) and \((3.10)\), we find that \( u_0 \) and \( v_0 \) are respectively lower and upper solutions of \((3.14)-(3.15)\), that is,
\[
u''_0(t) \geq N(t,u_0(t)) \geq B(t,u_0;u_0,v_0), \quad \alpha u_0(0) - \beta u'_0(0) \leq g_1(u_0(\sigma)) \leq \hat{h}_1(u_0(\sigma);v_0), 
\]
\[
\alpha u_0(1) + \beta u'_0(1) \leq g_2(u_0(\sigma)) \leq \hat{h}_2(u_0(\sigma);v_0), 
\]
and
\[
v''_0(t) \leq N(t,v_0(t)) = B(t,v_0;u_0,v_0), \quad \alpha v_0(0) - \beta v'_0(0) \geq g_1(v_0(\sigma)) = \hat{h}_1(v_0(\sigma);v_0), 
\]
\[
\alpha v_0(1) + \beta v'_0(1) \geq g_2(v_0(\sigma)) = \hat{h}_2(v_0(\sigma);v_0). 
\]
Again, it follows by Theorems 2.1 and 2.2 that there exists a unique solution \( v_1 \) of \((3.14)-(3.15)\) such that
\[
u_0(t) \leq v_1(t) \leq v_0(t), \quad t \in J. 
\] (3.16)
In order to show that \( u_1(t) \leq v_1(t) \), we need to prove that \( u_1(t) \) is a lower solution and \( v_1(t) \) is an upper solution of \((1.1)-(1.2)\).
Using the fact that \( u_1(t) \) is a solution of \((3.11)-(3.12)\) satisfying \( u_0(t) \leq u_1(t) \leq v_0(t) \) and \((3.7)-(3.8)\), we obtain
\[
u''_1(t) = A(t,u_1;u_0,v_0) \geq N(t,u_1(t)), 
\]
\[
\alpha u_1(0) - \beta u'_1(0) = h_i(u_1(\sigma);u_0,v_0) \leq g_1(u_1(\sigma)), 
\]
\[
\alpha u_1(1) + \beta u'_1(1) = h_i(u_1(\sigma);u_0,v_0) \leq g_2(u_1(\sigma)), 
\]
which shows that \( u_1 \) is a lower solution of \((1.1)-(1.2)\).
Since \( v_1(t) \) is a solution of \((3.14)-(3.15)\), it follows by using \((3.9)\) and \((3.10)\) that
\[
v''_1(t) = B(t,v_1;u_0,v_0) \leq N(t,v_1(t)), 
\]
Thus, $v_1$ is an upper solution of (1.1)-(1.2). Hence, by Theorem 2.1, we conclude that

$$u_1(t) \leq v_1(t), \quad t \in J. \tag{3.17}$$

Combining (3.13), (3.16) and (3.17) yields

$$u_0(t) \leq u_1(t) \leq v_1(t) \leq v_0(t), \quad t \in J.$$

Now, by induction, we prove that

$$u_0(t) \leq u_1(t) \leq \ldots \leq u_n(t) \leq v_n(t) \leq u_{n+1}(t) \leq v_{n+1}(t) \leq \ldots \leq v_1(t) \leq v_0(t).$$

For that, we consider the BVP’s

$$x''(t) = A(t, x; u_n, v_n), \quad t \in J, \tag{3.18}$$

$$\alpha x(0) - \beta x'(0) = h_1(x(\sigma); u_n, v_n), \quad \alpha x(1) + \beta x'(1) = h_2(x(\sigma); u_n, v_n),$$

and

$$x''(t) = B(t, x; u_n), \quad t \in J, \tag{3.19}$$

$$\alpha x(0) - \beta x'(0) = \hat{h}_1(x(\sigma); u_n), \quad \alpha x(1) + \beta x'(1) = \hat{h}_2(x(\sigma); u_n). \tag{3.21}$$

Assume that for some $n > 1$, $u_0(t) \leq u_n(t) \leq v_n(t) \leq v_0(t)$ and we will show that $u_0(t) \leq u_{n+1}(t) \leq v_{n+1}(t) \leq v_0(t)$.

Using (3.7) together with the fact that $u_n$ is the solution of (3.18)-(3.19), we have

$$u''_n(t) = A(t, u_n; u_{n-1}, v_{n-1}) \geq N(t, u_n) = A(t, u_n; u_n, v_n).$$

By the definition of $h_i$ and (3.8), we obtain

$$h_i(u_n(\sigma); u_{n-1}, v_{n-1}) \leq g_i(u_n(\sigma)) = h_i(u_n(\sigma); u_n, v_n),$$

which yields

$$\alpha u_n(0) - \beta u'_n(0) \leq h_1(u_n(\sigma); u_n, v_n), \quad \alpha u_n(1) + \beta u'_n(1) \leq h_2(u_n(\sigma); u_n, v_n).$$

Thus, $u_n$ is a lower solution of (3.18)-(3.19). On the same pattern, we find that $v_n$ is an upper solution of (3.18)-(3.19). Thus, by Theorems 2.1 and 2.2, there exists a unique solution $u_{n+1}(t)$ of (3.18)-(3.19) such that $u_n(t) \leq u_{n+1}(t) \leq v_{n+1}(t) \leq v_n(t)$.
\( v_n(t), \; t \in J \). Similarly it can be shown that there exists a unique solution \( v_{n+1}(t) \) of (3.20)-(3.21) such that \( u_n(t) \leq v_{n+1}(t) \leq v_n(t), \; t \in J \), where \( u_n(t) \) and \( v_n(t) \) are lower and upper solutions of (3.20)-(3.21) respectively.

Next, we establish the quadratic convergence of the sequences \( \{u_n\} \) and \( \{v_n\} \). We get

\[
\begin{align*}
\alpha u_{n+1}(0) - \beta u'_{n+1}(0) &= h_i(u_{n+1}(\sigma); u_n, v_n) \\
\alpha u_{n+1}(1) + \beta u'_{n+1}(1) &= h_i(u_{n+1}(\sigma); u_n, v_n)
\end{align*}
\]

which implies that \( u_{n+1} \) is a lower solution of (1.1)-(1.2). Analogously, it can be proved that \( v_{n+1} \) is an upper solution of (1.1)-(1.2). Hence, by Theorem 2.1, it follows that \( u_{n+1}(t) \leq v_{n+1}(t) \). Therefore, by induction, we have

\[
u_0(t) \leq u_1(t) \leq ... \leq u_n(t) \leq v_{n+1}(t) \leq v_n(t) \leq ... \leq v_1(t) \leq v_0(t), \; \forall n \in \mathbb{N},
\]

where \( u_n(t) \) and \( v_n(t) \) are given by

\[
\begin{align*}
u_n(t) &= \frac{1}{\alpha^2 + 2\alpha \beta} [h_1(u_n(\sigma); u_{n-1}, v_{n-1}) (\beta + \alpha (1 - t)) \\
&+ h_2(u_n(\sigma); u_{n-1}, v_{n-1}) (\beta + \alpha t)] + \int_0^1 G(t, s) A(s, u_n(s); u_{n-1}, v_{n-1}) ds,
\end{align*}
\]

and

\[
\begin{align*}
u_n(t) &= \frac{1}{\alpha^2 + 2\alpha \beta} [\hat{h}_1(v_n(\sigma); v_{n-1}) (\beta + \alpha (1 - t)) + \hat{h}_2(v_n(\sigma); v_{n-1}) (\beta + \alpha t)] \\
&+ \int_0^1 G(t, s) B(s, v_n(s); u_{n-1}, v_{n-1}) ds.
\end{align*}
\]

Since \([0, 1]\) is compact and the monotone convergence is pointwise, it follows that the convergence of each sequence \( \{u_n\} \) and \( \{v_n\} \) is uniform [10]. Suppose that \( x \) is a limit point of each of the sequences. Then, by taking the limit \( n \to \infty \), we obtain

\[
x(t) = \left( \frac{1}{\alpha^2 + 2\alpha \beta} \right) [g_1(x(\sigma)) (\beta + \alpha (1-t)) + g_2(x(\sigma)) (\beta + \alpha t)] + \int_0^1 G(t, s) N(s, x(s)) ds.
\]

This proves that the BVP (1.1)-(1.2) has the unique solution.

Now, we establish the quadratic convergence of the sequences \( \{u_n\} \) and \( \{v_n\} \). For that, we set \( q_n(t) = v_n(t) - x(t) \) and \( p_n(t) = x(t) - u_n(t) \), and note that
$q_n \geq 0$, $p_n \geq 0$. We will only prove the quadratic convergence of the sequence \{p_n\} as that of \{q_n\} is similar. Using the mean value theorem, we can find $u_n \leq \xi_1, \xi_2, \xi_3, \xi_4 \leq x$ so that

$$
p_{n+1}''(t) = x''(t) - u_{n+1}''(t)
= N(t, x) - A(t, u_{n+1}, u_n, v_n)
= F(t, x) + K(t, x) + H(t, x) - \phi(t, x) - \chi(t, x)
- F(t, u_n) - K(t, u_n) - H(t, u_{n+1}) + \phi(t, u_n) + \chi(t, u_n)
- [F_x(t, v_n) + K_x(t, u_n) - \phi_x(t, u_n) - \chi_x(t, v_n)](u_{n+1} - u_n)
\geq F_x(t, \xi_1)p_n + K_x(t, \xi_2)p_n - Lp_{n+1} - \phi_x(t, \xi_3)p_n - \chi_x(t, \xi_4)p_n
- [F_x(t, v_n) + K_x(t, u_n) - \phi_x(t, u_n) - \chi_x(t, v_n)](p_n - p_{n+1})$$

\[
\geq [F_x(t, u_n) - F_x(t, v_n)]p_n + [K_x(t, x) - K_x(t, u_n)]p_n
- [\phi_x(t, x) - \phi_x(t, u_n)]p_n + [\chi_x(t, v_n) - \chi_x(t, u_n)]p_n
+ [-L + F_x(t, v_n) + K_x(t, u_n) - \phi_x(t, u_n) - \chi_x(t, v_n)]p_{n+1}
\geq [-F_{xx}(t, \xi_5) + \chi_{xx}(t, \xi_8)]p_n(v_n - u_n) + K_{xx}(t, \xi_6)p_n^2 - \phi_{xx}(t, \xi_7)p_n^2
+ [-L + F_x(t, u_n) + K_x(t, u_n) - \phi_x(t, u_n) - \chi_x(t, u_n)]p_{n+1}
\geq -F_{xx}(t, \xi_5)p_n(q_n + p_n) + K_{xx}(t, \xi_6)p_n^2 - \phi_{xx}(t, \xi_7)p_n^2
+ \chi_{xx}(t, \xi_8)p_n(q_n + p_n)
\geq -\frac{3}{2}F_{xx}(t, \xi_5) - \frac{3}{2}\chi_{xx}(t, \xi_8) - K_{xx}(t, \xi_6) + \phi_{xx}(t, \xi_7)p_n^2
- \frac{1}{2}[F_{xx}(t, \xi_5) - \chi_{xx}(t, \xi_8)]q_n^2
\geq -(M_1\|p_n\|^2 + M_2\|q_n\|^2),
\]
where \( u_n \leq \zeta_5, \zeta_8 \leq v_n, u_n \leq \zeta_6, \zeta_7 \leq x, |F_{xx}| \leq A, |K_{xx}| \leq B, |\phi_{xx}| \leq C, |\chi_{xx}| \leq D \) on \( J \) and \( M_1 = \frac{3}{2} (A + D) + B + C, M_2 = D + A \). Thus,

\[
p_{n+1}(t) = x(t) - u_{n+1}(t)
\]

\[
= \frac{1}{\alpha^2 + 2\alpha\beta}[g_1(x(\sigma))(\beta + \alpha(1 - t)) + g_2(x(\sigma))(\beta + \alpha t)]
\]

\[
+ \int_0^1 G(t, s) N(s, x(s)) ds - \frac{1}{\alpha^2 + 2\alpha\beta} [h_1(u_{n+1}(\sigma); u_n, v_n)(\beta + \alpha(1 - t))]
\]

\[
+ h_2(u_{n+1}(\sigma); u_n, v_n)(\beta + \alpha t)] + \int_0^1 G(t, s) M(s, u_{n+1}(s); u_n, v_n) ds
\]

\[
= \frac{1}{\alpha^2 + 2\alpha\beta}[(g_1(x(\sigma)) - h_1(u_{n+1}(\sigma); u_n, v_n))(\beta + \alpha(1 - t)) + (g_2(x(\sigma)) - h_2(u_{n+1}(\sigma); u_n, v_n))(\beta + \alpha t)]
\]

\[
+ (M_1\|p_n\|^2 + M_2\|q_n\|^2) \int_0^1 |G(t, s)| ds
\]

\[
\leq \frac{1}{\alpha^2 + 2\alpha\beta}[(G'_1(\gamma_1) - \psi'_1(\gamma_1))p_n - G'_1(v_n(\sigma))(p_n(\sigma) - p_{n+1}(\sigma))
\]

\[
+ \psi'_1(\gamma_2)(p_n(\sigma) - p_{n+1}(\sigma))) (\beta + \alpha(1 - t)) + \{(G'_2(\delta_1) - \psi'_2(\delta_1))p_n(\sigma)
\]

\[
- G'_2(v_n(\sigma))(p_n(\sigma) - p_{n+1}(\sigma)) + \psi'_2(\delta_2)(p_n(\sigma) - p_{n+1}(\sigma))) (\beta + \alpha t)]
\]

\[
+ M_0(M_1\|p_n\|^2 + M_2\|q_n\|^2)
\]

\[
\leq \frac{1}{\alpha^2 + 2\alpha\beta}[(G'_1(\gamma_1) - \psi'_1(\gamma_1))p_n - G'_1(v_n(\sigma))(p_n(\sigma) - p_{n+1}(\sigma))
\]

\[
+ (G'_1(v_n(\sigma)) - \psi'_1(\gamma_1))p_{n+1}(\sigma))(\beta + \alpha(1 - t))
\]

\[
+ \{(G'_2(\gamma_1) - \psi'_2(\gamma_1))p_n(\sigma) - (G'_2(v_n(\sigma)) - \psi'_2(\gamma_2))(p_n(\sigma) - p_{n+1}(\sigma))\} (\beta + \alpha t)] + M_0(M_1\|p_n\|^2 + M_2\|q_n\|^2)
\]

\[
\leq \frac{1}{\alpha^2 + 2\alpha\beta}[-\frac{3}{2} G''_1(\rho_1) + \psi''_1(\rho_2)) p_n(\sigma) - \frac{1}{2} G''_1(\rho_1)\|q_n\|^2(\sigma)
\]

\[
+ \frac{1}{2} G''_2(\sigma_1)(\beta + \alpha(1 - t)) + \{(G''_2(\delta_2) - \psi''_2(\delta_2)) p_n(\sigma) - \frac{3}{2} G''_2(\sigma_2)\|q_n\|^2(\sigma)
\]

\[
- \frac{1}{2} G''_2(\sigma_1)\|q_n\|^2 + g'_1(v_n(\sigma))p_{n+1}(\sigma)) (\beta + \alpha t)] + M_0(M_1\|p_n\|^2 + M_2\|q_n\|^2),
\]
where \( u_n \leq \gamma_1, \delta_1, \rho_1, \sigma_1 \leq x, \ u_n \leq \gamma_2 \leq x, \) and \( u_n \leq \delta_2, \rho_2, \sigma_2 \leq u_{n+1}. \) Letting \( |G''_i| < A_i, \ |\psi''_i| < B_i, \ |g'_i| < C_i \) and taking the maximum over the interval \([0,1]\), we obtain

\[
\|p_{n+1}\| = \omega_1 \|p_n\|^2 + \omega_2 \|q_n\|^2 + \omega_3 \|p_{n+1}\|, \tag{3.22}
\]

where \( \left[\frac{1}{2}(A_1 + A_2) + B_1 + B_2 + M_0M_1\right] \eta_1 = \omega_1, \ \left[\frac{1}{2}(A_1 + A_2) + M_0M_2\right] \eta_1 = \omega_2, \ (C_1 + C_2) \eta_1 = \omega_3(< 1), \ \eta_1 = \frac{\beta + \alpha}{\alpha^2 + 2\alpha \beta}. \) Solving (3.22) algebraically, we have

\[
\|p_{n+1}\| = \frac{1}{1 - \omega_3} (\omega_1 \|p_n\|^2 + \omega_2 \|q_n\|^2).
\]

This proves the quadratic convergence.

4 Concluding Remarks.

A quasilinearization technique is applied to obtain monotone sequences of lower and upper solutions converging quadratically to the unique solution of a second order ordinary differential equation involving mixed nonlinearities with nonlinear nonlocal three-point boundary conditions. In fact, the existence of the unique solution of the boundary value problem is established by relaxing the convexity/concavity assumptions on the nonlinear functions present in the differential equation and the boundary conditions. The results obtained in this paper are new and yield some interesting special cases. If we take \( g_1(x(\sigma)) = a \) and \( g_2(x(\sigma)) = b \) (\( a \) and \( b \) are constants) in (1.2), our problem corresponds to a two-point boundary value problem involving mixed nonlinearities with separated boundary conditions. By fixing \( g_1(x(\sigma)) = a, \) we mean that the controller at the point \( t = 0 \) is kept at constant rate while the end \( t = 1 \) is dissipating or adding energy according to the censor located at \( t = \sigma \) (similarly, the end \( t = 1 \) can be kept at constant rate by taking \( g_2(x(\sigma)) = b \)). In case we take a specific value of \( \sigma, \) for instance \( \sigma = 1/2, \) our results correspond to a three-point local boundary value problem with mixed nonlinearities. If we choose \( \beta = 0, \) our problem takes the form of a three-point nonlinear nonlocal Dirichlet boundary value problem. By taking \( \beta = 0, \ \sigma = 1/2, \ \alpha = 1, \ g_1(x(\sigma)) = a, \ k(t, x) = 0, \ H(t, x) = 0 \) and \( \phi = 0 \) (in \((A_2), \) the results of [10] appear as a special case of our results. Moreover, our results generalize the work presented in [13] and [18].
References


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