Applied Mathematical Sciences, Vol. 4, 2010, no. 4, 145 - 175

A New Approach to Some Sobolev Injections in $\mathbb{R}^n$ Parallelepiped

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Abstract

Using eigenvalues and eigenfunctions of $-\Delta$ with homogeneous Dirichlet boundary conditions on $\partial\Omega$, where Ω is a parallelepiped of $\mathbb{R}^n$, we prove some Sobolev injections.

Mathematics Subject Classification: 35A35, 35J05, 45C05, 46E35

Keywords: Sobolev injections. parallelepiped. Eigenvalues and eigenfunctions. Poisson's operator

1.1 Introduction

The importance of the Sobolev inequalities in the theory of partial differential equations is well-established, and over the years much effort has been devoted to study of these inequalities. Let us recall that this type of inequalities arises in many questions of mathematical analysis (existence [8], [11], uniqueness

[25], [26], regularity [14], estimations [15], [19],[20], behaviour asymptotic [6], [16],[18],[24]...)
Homogeinity argument in [7] and Fourier transformation in [5] are used to establish Sobolev inequalities. In the present paper we deal with the injection

$$H_0^1(\Omega) \subset L^{2^*}(\Omega) \tag{1.1.1}$$

in the context to give a new proof. Here Ω is a domain of $\mathbb{R}^n$ $(n \geq 2)$ and $2^* = \frac{2n}{n-2}$ is the critical Sobolev exponent.
The method we present here is based on the constatation that every function in $H_0^1(\Omega)$ will be written as

$$\sum_j \alpha_j w_j$$

where w_j is a non trivial solution of the problem

$$\begin{cases} -\Delta w_j &= \lambda_j w_j \quad \text{on} \quad \Omega \\ w_j &= 0 \quad\quad \text{on} \quad \partial\Omega \end{cases}$$

Indeed (see [10], [23]), we know that the non negative sequence λ_j is non decreasing and eigenfunctions w_j form a total basis of $H_0^1(\Omega)$ in such away that:

$$\sum_j \alpha_j w_j \in H_0^1(\Omega) \Longleftrightarrow \sum_j \lambda_j \alpha_j^2 < +\infty$$

The explicit determination of the eigenvalues and eigenfunctions of $-\Delta$ with zero Dirichlet conditions on Ω, in general out of reach, depend on the geometry of Ω. If we consider some particular domains for which the geometry is sufficient simple, the computation is possible stressing on the methods based on the invariance of domain by a group of transformations (see [10]). In the present work we limit our analysis to study (1.1.1) in the case where Ω is a parallelepiped. Exploiting this, we take out the expression of eigenvalues and eigenfunctions of $-\Delta$ with zero Dirichlet conditions on Ω.
This paper consists of four sections. The first one is devoted to study (1.1.1) for $n \geq 5$. We will touch only a few aspects of the theory. In the second section, we will be concerned with the fourth dimension case. It contains a brief summary of some adapted results and provides a description of detailed proofs missing in section 1. Section 3 deals with the case $n = 3$. For the convenience of the reader we repeat the relevant materiel from sectin 2 without all the proofs, thus making our exposition self-contained. Lastly, in section 4, we indicate how these techniques may be used to examine the case $n = 2$. We will restrict our attention to prove

$$H^{\frac{1}{2}}(\Omega) \subset L^4(\Omega)$$

and

$$H^{\frac{2}{3}}(\Omega) \subset L^6(\Omega)$$

Our viewpoint sheds some new light on the injection

$$H^{1-\frac{1}{p}}(\Omega) \subset L^{2p}(\Omega) \text{ for all } p \in \mathbb{N}^* - \{1\}$$

Thus, to sum up, the motivation of this new approach in the study is threefold. Firstly, Sobolev spaces are fundamental tools in many pratical problems (control, PDE,...) and independently of that, they are interest. Second, the fact that the norm in $H_0^1(\Omega)$ is a discrete sum, will be taken advantage in numerical computations (see [2]). In addition, these new proofs will be turned on other research horizons as interpolation inequalities, weighted Sobolev inequalities, Poincare inequality....
It would be desirable to extend our argument to other domains in $\mathbb{R}^n$ (ball, annulus, cylinder...). Some subjects mentioned above are under investigation. In particular, it know that we have in the habit to looking for the asymptotic behavior of $N(\lambda)$, as $\lambda \to +\infty$, where $N(\lambda)$ is the number of eigenvalues $< \lambda$ of the Dirichlet Laplacian on an arbitrary bounded open set $\Omega \subset \mathbb{R}^n$ $(n \geq 1)$ with boundary $\partial\Omega$. Weyl's classical asymptotic formula [21] states that

$$N(\lambda) \sim c(n, \Omega)\, \lambda^{\frac{n}{2}}, \text{ as } \lambda \to +\infty$$

The standard method to value $N(\lambda)$ use the min-max form [9]. The basic research for regular Ω is due to L.Garding [13]. Later J.Fleckinger and G.Metivier [12] prove the same estimate for less regular bounded open set $\Omega \subset \mathbb{R}^n$. We note that there is a relation between the dimension n, the asymptotic behavior of $N(\lambda)$ and Sobolev inequalities. For the bounded open set $\Omega \subset \mathbb{R}^3$ defined in [1], [3] and [4], we hope a result as

$$H_0^1(\Omega) \subset L^p(\Omega) \text{ implies } N(\lambda) \sim c(\Omega)\, \lambda^{\frac{n(p)}{2}}, \text{ as } \lambda \to +\infty$$

or a reciprocal result. Clearly we don't expect that $n(p) = 3$.

2.2 The methodology

2.1 Description of the approach

Let $n \geq 3$ be any integer and Ω denote a parallelepiped of $\mathbb{R}^n$. Without loss of generality we can assume that $\Omega =]0,1[^n$. The eigenfunctions and the eigenvalues of $-\Delta$ with zero Dirichlet conditions on Ω are respectively defined by

$$w_j(x) = \prod_{r=1}^{n} \sin j_r \pi x_r \tag{2.2.2}$$

and

$$\lambda_j = \pi^2 |j|^2 \tag{2.2.3}$$

where

$$x = (x_1, x_2, \dots, x_n) \in \Omega,\ j = (j_1, j_2, \dots, j_n) \in (\mathbb{N}^*)^n \text{ and } |j|^2 = \sum_{r=1}^{n} j_r^2.$$

We know (see for example [10]) that if $u \in H_0^1(\Omega)$ then for all $j \in (\mathbb{N}^*)^n$, there exists $\alpha_j \in \mathbb{R}$ such that

$$u = \sum_j \alpha_j w_j \text{ and consequently } \sum_j \alpha_j^2 |j|^2 < +\infty.$$

Our idea is to establish, using the expression of w_j, that

$$\int_\Omega \left| \sum_j \alpha_j w_j \right|^{2^*} dx \le c \left(\sum_j \alpha_j^2 |j|^2 \right)^{\dfrac{2^*}{2}},$$

where c is a positive constant.
Throughout this paper, we may assume α_j to be positive. Hölder's inequality implies that

$$\begin{aligned}
&\int_\Omega \left| \sum_j \alpha_j w_j \right|^{2^*} dx \\
&\le \left[\int_\Omega \sum_{j^1, j^2, \dots, j^{2n} \in (\mathbb{N}^*)^n} \alpha_{j^1}\alpha_{j^2}\dots\alpha_{j^{2n}} w_{j^1} w_{j^2} \dots w_{j^{2n}} dx \right]^{\dfrac{1}{n-2}} \\
&\le \left[\sum_{j^1, j^2, \dots, j^{2n} \in (\mathbb{N}^*)^n} \alpha_{j^1}\alpha_{j^2}\dots\alpha_{j^{2n}} \prod_{k=1}^{n} \int_0^1 \prod_{r=1}^{2n} \sin j_k^r \pi y dy \right]^{\dfrac{1}{n-2}} \\
&\le c_1 \left[\sum_{\substack{j^1, j^2, \dots, j^{2n} \in (\mathbb{N}^*)^n \\ \epsilon^1, \epsilon^2, \dots, \epsilon^{2n} \in \{-1,1\}^n \\ \epsilon^1 j^1 + \epsilon^2 j^2 + \dots + \epsilon^{2n} j^{2n} = 0}} \left(\prod_{k=1}^{n} \epsilon_k^1 \epsilon_k^2 \dots \epsilon_k^{2n} \right) \prod_{k=1}^{2n} \alpha_{j^k} \right]^{\dfrac{1}{n-2}}
\end{aligned}$$

Throughout the proofs, $\epsilon^i j^i$ denotes

$$\left(\epsilon_1^i j_1^i, \epsilon_2^i j_2^i, \ldots, \epsilon_n^i j_n^i\right) \in \mathbb{Z}^n \text{ for } i = 1, 2, \ldots, 2n.$$

To make reading easier, we write

$$\beta\left(\epsilon^1, \epsilon^2, \ldots, \epsilon^{2n}\right) = \sum_{\substack{j^1, j^2, \ldots, j^{2n} \in (\mathbb{N}^*)^n \\ \epsilon^1 j^1 + \epsilon^2 j^2 + \cdots + \epsilon^{2n} j^{2n} = 0}} \alpha_{j^1} \alpha_{j^2} \ldots \alpha_{j^{2n}}. \tag{2.2.4}$$

With this notation, we have

$$\int_\Omega \left|\sum_j \alpha_j w_j\right|^{2^*} dx \leq c_1 \left[\sum_{\epsilon^1, \epsilon^2, \ldots, \epsilon^{2n} \in \{-1,1\}^n} \left(\prod_{k=1}^n \epsilon_k^1 \epsilon_k^2 \ldots \epsilon_k^{2n}\right) \beta\left(\epsilon^1, \epsilon^2, \ldots, \epsilon^{2n}\right)\right]^{\frac{1}{n-2}}.$$

2.2 First estimates

We first to estimate $\beta\left(\epsilon^1, \epsilon^2, \ldots, \epsilon^{2n}\right)$ for particular choice of ϵ^i with $i \in \{1, \ldots, 2n\}$.

Lemme 2.2.1 *Suppose that*

$$\epsilon^1 = \epsilon^2 = \cdots = \epsilon^n = -\epsilon^{n+1} = -\epsilon^{n+2} = \cdots = -\epsilon^{2n}.$$

Then we have

$$\beta_{0\epsilon} \leq c_2 \left(\sum_j \alpha_j^2 |j|^2\right)^n \tag{2.2.5}$$

where $\beta_{0\epsilon} = \beta(\epsilon, \epsilon, \ldots, \epsilon, -\epsilon, -\epsilon, \ldots, -\epsilon)$ *for all* $\epsilon \in \{-1,1\}^n$ *and* c_2 *is a positive constant.*

Proof - From (2.2.4) it follows that

$$\begin{aligned} \beta_{0\epsilon} &= \sum_{j^1 + j^2 + \cdots + j^n = j^{n+1} + j^{n+2} + \cdots + j^{2n}} \alpha_{j^1} \alpha_{j^2} \ldots \alpha_{j^{2n}} \\ &= \sum_L \left(\sum_{j^1 + j^2 + \cdots + j^n = L} \alpha_{j^1} \alpha_{j^2} \ldots \alpha_{j^n}\right)^2. \end{aligned}$$

Using Cauchy-Schwarz inequality, we obtain

$$\left(\sum_{j^1+j^2+\cdots+j^n=L}\alpha_{j^1}\alpha_{j^2}\ldots\alpha_{j^n}\right)^2 \le \left(\sum_{j^1+j^2+\cdots+j^n=L}\prod_{k=1}^{n}\alpha_{j^k}^2\left|j^k\right|^2\right) \times\left(\sum_{j^1+j^2+\cdots+j^n=L}\frac{1}{|j^1|^2|j^2|^2\ldots|j^n|^2}\right). \tag{2.2.6}$$

Our next claim is that

$$\sum_{j^1+j^2+\cdots+j^n=L}\frac{1}{|j^1|^2|j^2|^2\ldots|j^n|^2}\le m$$

where m is independent of L. Indeed

$$\begin{aligned}\sum_{j^1+j^2+\cdots+j^n=L}\frac{1}{|j^1|^2|j^2|^2\ldots|j^n|^2} &\le 2^n\sum_{\substack{j^1+j^2+\cdots+j^n=L\\ |j^1|\ge|j^2|\ge\cdots\ge|j^n|}}\frac{1}{|j^1|^2|j^2|^2\ldots|j^n|^2}\\ &\le \frac{2^n n^2}{|L|^2}\sum_{j^n=I\!I_n}^{L}\frac{1}{|j^n|^{2(n-1)}}\end{aligned}$$

where $I\!I_n=(1,1,\ldots,1)\in I\!N^n$.
Comparing sum to integral we can assert that (see [17], page 103)

$$\int_{\sqrt{n}<|x|<|L|}\frac{dx}{|x|^{2(n-1)}}\le m_1 \tag{2.2.7}$$

where the constant m_1 depends on the measure of the $(n-1)$-dimensional unit sphere in $I\!R^n$.
Combining (2.2.6) and (2.2.7) we obtain that

$$\begin{aligned}\sum_L\left(\sum_{j^1+j^2+\cdots+j^n=L}\alpha_{j^1}\alpha_{j^2}\ldots\alpha_{j^n}\right)^2 &\le \sum_L m\left(\sum_{j^1+j^2+\cdots+j^n=L}\prod_{k=1}^{n}\alpha_{j^k}^2\left|j^k\right|^2\right)\\ &\le m\left(\sum_j\alpha_j^2|j|^2\right)^n.\end{aligned}$$

■

Corollaire 2.2.1 *Under the assumptions of the above lemma with*

$$\epsilon^1 = \epsilon^2 = \cdots = \epsilon^n = -\epsilon^{n+1} = -\epsilon^{n+2} = \cdots = -\epsilon^{2n}$$

replaced by

$$\epsilon^1 = \epsilon^2 = \cdots = \epsilon^{n+r} = -\epsilon^{n+r+1} = -\epsilon^{n+2} = \cdots = -\epsilon^{2n}\,(r \in \{1, 2, \ldots, n-1\})$$

we have

$$\beta_{r\epsilon} \leq c_1 \left(\sum_j \alpha_j^2 \left|j\right|^2 \right)^n \tag{2.2.8}$$

where

$$\begin{aligned} \beta_{r\epsilon} &= \beta\left(\epsilon^1, \ldots, \epsilon^n, \ldots, \epsilon^{n+r}, \epsilon^{n+r+1}, \ldots, \epsilon^{2n}\right) \\ &= \beta\left(\epsilon, \ldots, \epsilon, \ldots, \epsilon, -\epsilon, \ldots, -\epsilon\right) \textit{ for all } \epsilon \in \{-1, 1\}^n. \end{aligned}$$

Proof - Let $r \in \{1, 2, \ldots, n-1\}$. Using (2.2.4), we have

$$\beta_{r\epsilon} = \sum_{j^1+j^2+\cdots+j^{n+r}=j^{n+r+1}+\cdots+j^{2n}} \alpha_{j^1}\alpha_{j^2}\ldots\alpha_{j^{2n}}. \tag{2.2.9}$$

Let us introduce the notation

$$\bar{j}^r = \sum_{k=n+r+1}^{2n} j^k$$

and

$$\underline{j}^r = \sum_{k=n+1}^{n+r} j^k.$$

For all $i \in \{1, 2, \ldots, n\}$ we have

$$\bar{j}_i^r - \underline{j}_i^r \geq n.$$

Then (2.2.9) becomes

$$\beta_{r\epsilon} = \sum_{j^{n+r+1},\ldots,j^{2n}} \sum_{j^{n+1},\ldots,j^{n+r}} \sum_{j^1+j^2+\cdots+j^n=\bar{j}^r\text{-}\underline{j}^r} \alpha_{j^1}\alpha_{j^2}\ldots\alpha_{j^{2n}}.. \tag{2.2.10}$$

Define

$$\gamma_L = \sum_{j^1+j^2+\cdots+j^n=L} \alpha_{j^1}\alpha_{j^2}\ldots\alpha_{j^n}. \tag{2.2.11}$$

Lemma 2.2.1 shows that

$$\sum_L \gamma_L^2 \leq c_2 \left(\sum_j \alpha_j^2 \, |j|^2 \right)^n .$$

Combining (2.2.10) with (2.2.11), we can assert that

$$\begin{aligned}
\beta_{r\epsilon} &\leq \sum_{j^{n+1}, \ldots, j^{n+r}} \; \sum_{j^{n+r+1}, \ldots, j^{2n}} \alpha_{j^{n+1}} \alpha_{j^{n+2}} \ldots \alpha_{j^{n+r}} \alpha_{j^{n+r+1}} \alpha_{j^{n+r+2}} \ldots \alpha_{j^{2n}} \gamma_{\bar{j}^r - \underline{j}^r} \\
&\leq \sum_{R, \, j^{n+1}, \ldots, j^{2n-1}} \alpha_{j^{n+1}} \alpha_{j^{n+2}} \ldots \alpha_{j^{2n}} \gamma_r
\end{aligned}$$

where

$$R = \bar{j}^r - \underline{j}^r$$

and consequently

$$j^{2n} = R - \sum_{i=n+r+1}^{2n-1} j^i + \underline{j}^r .$$

Cauchy-Schwarz inequality shows that

$$\begin{aligned}
\beta_{r\epsilon} &\leq \left(\sum_r |\gamma_r|^2 \right)^{\frac{1}{2}} \left(\sum_r \left(\sum_{j^{n+1}, \ldots, j^{2n-1}} \alpha_{j^{n+1}} \alpha_{j^{n+2}} \ldots \alpha_{j^{2n}} \right)^2 \right)^{\frac{1}{2}} \\
&\leq \left(\sum_r |\gamma_r|^2 \right)^{\frac{1}{2}} \tilde{\beta_0}^{\frac{1}{2}}
\end{aligned}$$

where

$$\begin{aligned}
\tilde{\beta_0} &= \sum_r \left(\sum_{j^{n+1}, \ldots, j^{2n-1}} \alpha_{j^{n+1}} \alpha_{j^{n+2}} \ldots \alpha_{j^{2n}} \right)^2 \\
&= \sum_{j^{n+1} + \cdots + j^{2n} = m^{n+1} + \cdots + m^{2n}} \prod_{i=n+1}^{2n} \alpha_{j^i} \prod_{i=n+1}^{2n} \alpha_{m^i} .
\end{aligned}$$

Lemma 2.2.1 implies that

$$\tilde{\beta_0} \leq c_2 \left(\sum_j \alpha_j^2 \, |j|^2 \right)^n$$

which completes the proof. ■

2.3 A general tool

Note that we have actually studied a particular choice of $\epsilon^1, \epsilon^2, \dots, \epsilon^{2n}$. The general constraint

$$\sum_{i=1}^{2n} \epsilon^i j^i = 0$$

can be written

$$\left\{ \begin{array}{lcl} j_1^{\sigma_1(1)} + j_1^{\sigma_1(2)} + \cdots + j_1^{\sigma_1(n+r_1)} & = & j_1^{\sigma_1(n+r_1+1)} + \cdots + j_1^{\sigma_1(2n)} \\ j_2^{\sigma_2(1)} + j_2^{\sigma_2(2)} + \cdots + j_2^{\sigma_2(n+r_2)} & = & j_2^{\sigma_2(n+r_2+1)} + \cdots + j_2^{\sigma_2(2n)} \\ & & \\ j_n^{\sigma_n(1)} + j_n^{\sigma_n(2)} + \cdots + j_n^{\sigma_n(n+r_n)} & = & j_n^{\sigma_n(n+r_n+1)} + \cdots + j_n^{\sigma_n(2n)} \end{array} \right. \qquad (C)$$

where $\sigma_1, \sigma_2, \dots, \sigma_n$ are permutations of {1,2, ... ,2n} and $n+r_l$ $(l = 1, 2, \dots, n)$ $(r_l \geq 0)$ is the number of the same signs at the line $\left(\sum_{i=1}^{2n} \epsilon_l^i j_l^i\right)$.

The main result in this subsection is the following

Lemme 2.2.2 *Assume that*

$$\sum_{(C_0)} \alpha_{j^1}\alpha_{j^2}\dots\alpha_{j^{2n}} \leq c_3 \left(\sum_j \alpha_j^2 |j|^2\right)^n \qquad (2.2.12)$$

where (C_0) *is the constraint* (C) *in which there* $r_1 = r_2 = \cdots = r_n = 0$. *Then*

$$\sum_{(C)} \alpha_{j^1}\alpha_{j^2}\dots\alpha_{j^{2n}} \leq c_3 \left(\sum_j \alpha_j^2 |j|^2\right)^n . \qquad (2.2.13)$$

Proof - There is no loss of generality in assuming that σ_1 is the identity permutation of {1,2, ... ,2n}. Then we can rewrite the constraint (C) as

$$\sum_{i=1}^{n} \eta^i j^i = \sum_{i=n+1}^{2n} \eta^i j^i$$

where

$$\eta^i \in \{1, -1\}^n \text{ for } 1 \leq i \leq 2n.$$

According to the above assumption, we have $\eta_1^i = 1$ for $i = 1, 2, \dots, n$. Thus

$$\sum_{(C)} \prod_{i=1}^{2n} \alpha_{j^i} =$$

$$\sum_{j^{n+1}, j^{n+2}, \dots, j^{2n}} \prod_{i=n+1}^{2n} \alpha_{j^i} \sum_{\eta^1 j^1 + \cdots + \eta^n j^n = \eta^{n+1} j^{n+1} + \cdots + \eta^{2n} j^{2n}} \prod_{i=1}^{n} \alpha_{j^i}.$$

For all $j \in \{1, 2, \ldots, n\}$, define

$$K_j = \left| \sum_{i=n+1}^{2n} \eta_j^i j_j^i \right|.$$

For all $j \in \{1, 2, \ldots, n\}$, we obtain

$$j_j^{2n} = \varepsilon \eta_j^{2n} K_j - \eta_j^{2n} \sum_{i=n+1}^{2n-1} \eta_j^i j_j^i$$

where

$$\varepsilon = \begin{cases} 1 & \text{if} \quad \sum_{i=n+1}^{2n} \eta_j^i j_j^i > 0 \\ -1 & \text{if} \quad \sum_{i=n+1}^{2n} \eta_j^i j_j^i < 0. \end{cases}$$

Therefore

$$\sum_{(C)} \prod_{i=1}^{2n} \alpha_{j^i} \leq \sum_{K, j^{n+1}, \ldots, j^{2n-1}} \alpha_{I\!K_j} \prod_{i=n+1}^{2n-1} \alpha_{j^i} \sum_{\eta^1 j^1 + \cdots + \eta^n j^n = \varepsilon K} \prod_{i=1}^{n} \alpha_{j^i}$$

where

$$K = (K_1, K_2, \ldots, K_n)$$

and

$$I\!K_j = \varepsilon \eta^{2n} K - \eta^{2n} \sum_{i=n+1}^{2n-1} \eta^i j^i.$$

Using Cauchy-Schwarz inequality, we obtain

$$\begin{aligned} \sum_{(C)} \prod_{i=1}^{2n} \alpha_{j^i} \quad \leq \quad & \leq \left(\sum_{K,\, j^{n+1}, \ldots, j^{2n-1}, l^{n+1}, \ldots, l^{2n-1}} \alpha_{I\!K_j} \prod_{i=n+1}^{2n-1} \alpha_{j^i} \ \alpha_{I\!K_l} \prod_{i=n+1}^{2n-1} \alpha_{l^i} \right)^{\frac{1}{2}} \\ & \times \left(\sum_K \left(\sum_{\eta^1 j^1 + \cdots + \eta^n j^n = \varepsilon K} \prod_{i=1}^{n} \alpha_{j^i} \right)^2 \right)^{\frac{1}{2}}. \end{aligned} \tag{2.2.14}$$

Note that

$$\sum_{i=n+1}^{2n-1} \eta^{2n} \eta^i j^i + I\!K_j = \sum_{i=n+1}^{2n-1} \eta^{2n} \eta^i l^i + I\!K_l \tag{2.2.15}$$

so that (2.2.14) may be written as

$$\sum_{(C)} \alpha_{j^1}\alpha_{j^2}\ldots\alpha_{j^{2n}} \le$$

$$\left(\sum_{\delta^1 m^1+\cdots+\delta^n m^n=\delta^1 m^{n+1}+\cdots+\delta^n m^{2n}} \prod_{i=1}^{2n}\alpha_{m^i}\right)^{\frac{1}{2}} \left(\sum_K \left(\sum_{\eta^1 j^1+\cdots+\eta^n j^n=\varepsilon K} \prod_{i=1}^{n}\alpha_{j^i}\right)^2\right)^{\frac{1}{2}}$$

where

- $\delta^i = \delta^{i+n} = \eta^{2n}\eta^i$ for $i = 1, 2, \ldots, n$,
- $m^i = j^i$ for $i = 1, 2, \ldots, n-1$,
- $m^i = l^{i-n}$ for $i = n+1, n+2, \ldots, 2n-1$,
- $m^n = I\!K_j$ and $m^{2n} = I\!K_l$.

According to (2.2.15) and (2.2.12), we conclude that

$$\sum_{(C)} \alpha_{j^1}\alpha_{j^2}\ldots\alpha_{j^{2n}} \le c_3 \left(\sum_j \alpha_j^2 \left|j\right|^2\right)^n$$

which is the desired conclusion. ■

Remarque 2.2.1 *The inequality* (2.2.12) *turns out to be rather technically sophisticated and we have a complete solution in the following particular case:* $n = 3$ *and* $n = 4$.

3.3 The case : $n = 4$

Thourought this section, we denote by Ω the parallelepiped $]0, 1[^4$ in $I\!R^4$. We recall that (see (2.2.2) and (2.2.3))

$$w_j(x) = \prod_{r=1}^{4} sin\left(j_r \pi x_r\right)$$

and

$$\lambda_j = \pi^2 \left|j\right|^2$$

are the eigenfunctions and eigenvalues respectively of $-\Delta$ with zero Dirichlet conditions on Ω. Here

$$j = (j_1, j_2, j_3, j_4) \in (\mathbb{N}^*)^4$$

and

$$x = (x_1, x_2, x_3, x_4) \in \Omega.$$

If $u = \sum_j \alpha_j w_j \in H_0^1(\Omega)$ then

$$\begin{aligned} \int_\Omega u^4 dx &= \int_\Omega \sum_{j^1, j^2, j^3, j^4 \in \left(\mathbb{N}^*\right)^4} \alpha_{j^1} \alpha_{j^2} \alpha_{j^3} \alpha_{j^4} w_{j^1} w_{j^2} w_{j^3} w_{j^4} dx \\ &= \tfrac{1}{8} \sum_{\epsilon^1, \epsilon^2, \epsilon^3, \epsilon^4 \in \{-1,1\}^4} \prod_{r=1}^4 \epsilon_r^1 \epsilon_r^2 \epsilon_r^3 \epsilon_r^4 \beta\left(\epsilon^1, \epsilon^2, \epsilon^3, \epsilon^4\right) \end{aligned}$$

where

$$\beta\left(\epsilon^1, \epsilon^2, \epsilon^3, \epsilon^4\right) = \sum_{\substack{j^1, j^2, j^3, j^4 \in (\mathbb{N}^*)^4 \\ \epsilon^1 j^1 + \epsilon^2 j^2 + \epsilon^3 j^3 + \epsilon^4 j^4 = 0}} \alpha_{j^1} \alpha_{j^2} \alpha_{j^3} \alpha_{j^4}.$$

We collect a few simple cases that we shall use in the sequel.

Lemme 3.3.1 *There exists $c_4 > 0$ such that for every $\epsilon \in \{-1, 1\}^4$*

$$\beta(\epsilon, \epsilon, -\epsilon, -\epsilon) \leq c_4 \left(\sum_j \alpha_j^2 |j|^2 \right)^2 \tag{3.3.1}$$

Without loss of generality, we fix $\epsilon^4 = (1, 1, 1, 1)$.

Lemme 3.3.2 *Let $\epsilon \in \{-1, 1\}^4$ and $L \in \mathbb{Z}^4$. If*

$$\sum_L \gamma_{L,\epsilon}^2 \leq c_5 \left(\sum_j \alpha_j^2 |j|^2 \right)^2 \tag{3.3.2}$$

where c_5 is a positive constant and $\gamma_{L,\epsilon} = \sum_j \alpha_j \alpha_{L-\epsilon j}$, then

$$\beta\left(\epsilon^1, \epsilon^2, \epsilon^3, \epsilon^4\right) \leq c_5 \left(\sum_j \alpha_j^2 |j|^2 \right)^2$$

for all $\epsilon^1, \epsilon^2, \epsilon^3, \epsilon^4 \in \{-1, 1\}^4$.

Proof - Indeed

$$\begin{aligned}\beta\left(\epsilon^1,\epsilon^2,\epsilon^3,\epsilon^4\right) &= \sum_{j^1,j^2}\alpha_{j^1}\alpha_{j^2}\sum_{\epsilon^3 j^3+\epsilon^4 j^4=-\epsilon^1 j^1-\epsilon^2 j^2}\alpha_{j^3}\alpha_{j^4}\\ &= \sum_{j^1,j^2,L}\alpha_{j^1}\alpha_{j^2}\gamma_{L,\epsilon^3\epsilon^4}\end{aligned}$$

where

$$L=-\epsilon^3\left(\epsilon^1 j^1+\epsilon^2 j^2\right).$$

Therefore

$$\begin{aligned}\beta\left(\epsilon^1,\epsilon^2,\epsilon^3,\epsilon^4\right) &= \sum_{L}\gamma_{L,\epsilon^3\epsilon^4}\sum_{j^1}\alpha_{j^1}\alpha_{-\epsilon^2\epsilon^3 L-\epsilon^1\epsilon^2 j^1}\\ &= \sum_{L}\gamma_{L,\epsilon^3\epsilon^4}\gamma_{-\epsilon^2\epsilon^3 L,\epsilon^1\epsilon^2}.\end{aligned}$$

Combining Cauchy-Schwarz inequality and (3.3.2), we obtain

$$\beta\left(\epsilon^1,\epsilon^2,\epsilon^3,\epsilon^4\right)\le c_5\left(\sum_j\alpha_j^2\left|j\right|^2\right)^2. \qquad \blacksquare$$

Remarque 3.3.1 *We may change the left hand side of the inequality* (3.3.1) *by*

$$\beta\left(\epsilon,-\epsilon,-\epsilon,-\epsilon\right).$$

Then the conclusion of lemma 3.3.1 remains true of course with the same proof as in corollary 2.2.1.

Remarque 3.3.2 *Obviously in the left hand side of inequality* (2.2.12), *we can have any choice of* $\epsilon^1,\epsilon^2,\epsilon^3$ *and* ϵ^4 *instead of particular choice proposed in lemma 3.3.1 and remark 3.3.1. The same estimate holds, but the proofs are technical. We shall discuss the remaining cases and see that we only need to consider two cases.*

Remarque 3.3.3 *The assumption* (3.3.2) *is proved in subsection 2.1 for* $\epsilon=(1,1,1,1)$ *and* $\epsilon=(-1,-1,-1,-1)$. *There remain essentially three other cases*

$$\epsilon=(1,1,1,-1),\ \epsilon=(1,1,-1,-1)\ \text{and}\ \epsilon=(1,-1,-1,-1).$$

Let

$$\epsilon^{01} = (1,1,1,-1)\,,\ \epsilon^{02} = (1,1,-1,-1) \text{ and } \epsilon^{03} = (1,1,-1,1)$$

To verify assumption (2.2.12) in the fourth dimension suffice it to estimate

- $\beta_1 = \beta\left(\epsilon^{01}, -\epsilon^4, -\epsilon^{01}, \epsilon^4\right)$;
- $\beta_2 = \beta\left(\epsilon^{02}, -\epsilon^4, -\epsilon^{02}, \epsilon^4\right)$;
- $\beta_3 = \beta\left(\epsilon^{02}, -\epsilon^{01}, -\epsilon^{03}, \epsilon^4\right)$;

after which we deduce the estimate of $\beta\left(\epsilon^1, \epsilon^2, \epsilon^3, \epsilon^4\right)$ for all ϵ^2 and ϵ^3.

3.1 Estimate of β_1

For

$$j = (j_1, j_2, j_3, j_4) \in (I\!N^*)^4$$

we write

$$\hat{j}_4 = (j_1, j_2, j_3)\,.$$

Lemme 3.3.3 *There exists* $c_6 > 0$ *such that*

$$\beta_1 \le c_6 \left(\sum_j \alpha_j^2 \,|j|^2\right)^2 .$$

Proof - We have

$$\begin{aligned}
\beta_1 &= \sum_{j+\epsilon^{01}k=l+\epsilon^{01}m} \alpha_j\alpha_k\alpha_l\alpha_m \\
&= \sum_{\hat{j}_4+\hat{k}_4=\hat{l}_4+\hat{m}_4}\ \sum_{j_4-k_4=l_4-m_4} \alpha_j\alpha_k\alpha_l\alpha_m \\
&= \sum_{\hat{L}_4, L_4} \left(\sum_{\hat{j}_4+\hat{k}_4=\hat{L}_4}\ \sum_{j_4-k_4=L_4} \alpha_j\alpha_k\right)^2 \\
&\le \sum_{\hat{L}_4, L_4} \left(\sum_{\hat{j}_4+\hat{k}_4=\hat{L}_4}\ \sum_{j_4-k_4=L_4} \alpha_j^2\,|j|^2\,\alpha_k^2|k|^2\right)\left(\sum_{\hat{j}_4+\hat{k}_4=\hat{L}_4}\ \sum_{j_4-k_4=L_4} \frac{1}{|j|^2\,|k|^2}\right).
\end{aligned}$$

Add to it, we have

$$\begin{aligned}
\sum_{j_4-k_4=L_4} \frac{1}{|j|^2|k|^2} &= \sum_{j_4-k_4=L_4} \frac{1}{\left(|\hat{j}_4|^2+j_4^2\right)\left(|\hat{k}_4|^2+k_4^2\right)} \\
&\le 2\sum_{j_4=1}^{+\infty} \frac{1}{\left(|\hat{j}_4|^2+j_4^2\right)\left(|\hat{k}_4|^2+j_4^2\right)} \\
&\le d_1 \int_1^{+\infty} \frac{dx}{\left(|\hat{j}_4|^2+x^2\right)\left(|\hat{k}_4|^2+x^2\right)} \\
&= d_1 \begin{cases} \dfrac{1}{|\hat{j}_4|^2-|\hat{k}_4|^2}\left(\dfrac{1}{1+k_1+k_2+k_3}-\dfrac{1}{1+j_1+j_2+j_3}\right) \\ \text{if } j_1+j_2+j_3 \neq k_1+k_2+k_3 \\ \dfrac{1}{3}\dfrac{1}{(1+j_1+j_2+j_3)^2} \text{ if } j_1+j_2+j_3 = k_1+k_2+k_3. \end{cases}
\end{aligned}$$

Therefore

$$\sum_{\hat{j}_4+\hat{k}_4=\hat{L}_4} \sum_{j_4-k_4=L_4} \frac{1}{|j|^2|k|^2}$$

$$\le d_2 \left(\frac{1}{\left|\hat{L}_4\right|} \sum_{\substack{\hat{j}_4+\hat{k}_4=\hat{L}_4 \\ \left|\hat{j}_4\right| \neq \left|\hat{k}_4\right|}} \frac{1}{\left(1+\left|\hat{j}_4\right|\right)\left(1+\left|\hat{k}_4\right|\right)} + \frac{\left(\left|\hat{L}_4\right|-2\right)\left(\left|\hat{L}_4\right|-4\right)\left(\left|\hat{L}_4\right|-5\right)}{3\left(8+\left|\hat{L}_4\right|\right)^3} \right)$$

$$\le d_2 \left(\frac{1}{3} + \frac{1}{\left|\hat{L}_4\right|} \sum_{\substack{\hat{j}_4+\hat{k}_4=\hat{L}_4 \\ \left|\hat{j}_4\right| \neq \left|\hat{k}_4\right|}} \frac{1}{\left(1+\left|\hat{j}_4\right|\right)\left(1+\left|\hat{k}_4\right|\right)} \right)$$

and then

$$\begin{aligned}
\sum_{\substack{\hat{j}_4+\hat{k}_4=\hat{L}_4\\ |\hat{j}_4|\neq|\hat{k}_4|}} \frac{1}{\left(1+|\hat{j}_4|\right)\left(1+|\hat{k}_4|\right)} &\leq 2\sum_{I\!I_3<\hat{j}_4<\hat{L}_4} \frac{1}{\left(1+|\hat{j}_4|\right)^2}\\
&\leq d_3\int_{4<|x|<|\hat{L}_4+I\!I_3|}\frac{dx}{|x|^2}\\
&= d_3\int_0^{\frac{\pi}{2}}\int_0^{\frac{\pi}{2}}\int_4^{|\hat{L}_4+I\!I_3|}\frac{r^2\sin\theta}{r^2}drd\theta d\varphi\\
&= d_3\frac{\pi}{2}\left(\left|\hat{L}_4+I\!I_3\right|-4\right).
\end{aligned}$$

Finally

$$\begin{aligned}
\beta_1 &\leq c_6\sum_L\left(\sum_{j+\epsilon^{01}k=L}\alpha_j^2\,|j|^2\,\alpha_k^2|k|^2\right)\\
&= c_6\left(\sum_j\alpha_j^2\,|j|^2\right)^2. \quad \blacksquare
\end{aligned}$$

3.2 Estimate of β_2

Lemme 3.3.4 *There exists $c_7 > 0$ such that*

$$\beta_2 \leq c_7\left(\sum_j\alpha_j^2\,|j|^2\right)^2. \tag{3.3.3}$$

Proof - Let us first prove that for all $L \in (I\!N-\{0,1\})^2 \times Z\!\!Z^2$, we have

$$f(L) = \sum_{\substack{j_1+k_1=L_1\\ j_2+k_2=L_2}}\ \sum_{\substack{j_3-k_3=L_3\\ j_4-k_4=L_4}} \frac{1}{|j|^2\,|k|^2} \leq c_7$$

where c_7 is being independent of L.
We have

$$\sum_{j_3-k_3=L_3}\ \sum_{j_4-k_4=L_4} \frac{1}{|j|^2\,|k|^2}$$

$$\leq d_4 \sum_{j_3=1}^{+\infty} \sum_{j_4=1}^{+\infty} \frac{1}{(j_1+j_2+j_3+j_4)^2\,(k_1+k_2+j_3+j_4)^2}$$

$$\leq d_5 \int_1^{+\infty}\int_1^{+\infty} \frac{dxdy}{(j_1+j_2+x+y)^2\,(k_1+k_2+x+y)^2}$$

$$\leq d_6 \int_0^{+\infty}\int_0^{+\infty} \frac{dxdy}{\left((j_1+j_2+2)^2+x^2+y^2\right)\left((k_1+k_2+2)^2+x^2+y^2\right)}$$

$$\leq d_6 \int_0^{+\infty}\int_0^{\frac{\pi}{2}} \frac{rdrd\theta}{\left((j_1+j_2+2)^2+r^2\right)\left((k_1+k_2+2)^2+r^2\right)}$$

$$\leq d_7 \begin{cases} \dfrac{1}{(k_1+k_2+2)^2-(j_1+j_2+2)^2}\ln\left(\dfrac{k_1+k_2+2}{j_1+j_2+2}\right) & \text{if}\quad j_1+j_2\neq k_1+k_2 \\ \dfrac{1}{(j_1+j_2+2)^2} & \text{if}\quad j_1+j_2=k_1+k_2. \end{cases}$$

If $(n_1,n_2)\in(I\!N^*)^2$, using $n_{12}=n_1+n_2$. Then

$$f(L)\leq$$

$$d_7\left(\sum_{\substack{(j_1,j_2)+(k_1,k_2)=(L_1,L_2)\\ j_{12}\neq k_{12}}} \frac{1}{(k_{12}+2)^2-(j_{12}+2)^2}\ln\left(\frac{k_{12}+2}{j_{12}+2}\right)+\frac{2(L_{12}-3)(L_{12}-2)}{(L_{12}+4)^2}.\right)$$

It is clear that

$$\frac{2\left(L_{12}-3\right)\left(L_{12}-2\right)}{\left(L_{12}+4\right)^2}\leq 2.$$

We now turn to the term

$$g\left(L_1,L_2\right)=\sum_{\substack{(j_1,j_2)+(k_1,k_2)=(L_1,L_2)\\ j_{12}\neq k_{12}}} \frac{1}{(k_{12}+2)^2-(j_{12}+2)^2}\ln\left(\frac{k_{12}+2}{j_{12}+2}\right).$$

We obtain with $j_{12} \neq \frac{1}{2}L_{12}$ that

$$\begin{aligned}
g\left(L_1, L_2\right) &= \sum_{j_1=1}^{L_1-1}\sum_{j_2=1}^{L_2-1} \frac{1}{\left(L_{12}-j_{12}+2\right)^2-\left(j_{12}+2\right)^2} \ln\left(\frac{L_{12}-j_{12}+2}{j_{12}+2}\right) \\
&= \frac{1}{L_{12}+4}\sum_{j_1=1}^{L_1-1}\sum_{j_2=1}^{L_2-1} \frac{1}{L_{12}-2\left(j_{12}\right)} \ln\left(\frac{L_{12}-j_{12}+2}{j_{12}+2}\right) \\
&\leq \frac{d_8}{L_{12}+4}\int_1^{L_1-1}\int_1^{L_2-1} \ln\left(\frac{L_{12}-x-y+2}{x+y+2}\right)\frac{dy}{L_{12}-2\left(x+y\right)}dx.
\end{aligned}$$

Let $u = x + y$ and $v = x$, then

$$\begin{cases} 1 < v < L_1 - 1 \\ 1 + v < u < L_2 - 1 + v. \end{cases}$$

Thus,

$$\begin{aligned}
g\left(L_1, L_2\right) &\leq \frac{d_8}{L_{12}+4}\int_1^{L_1-1}\int_{1+v}^{L_2-1+v} \ln\left(\frac{L_{12}-u+2}{u+2}\right)\frac{du}{L_{12}-2u}dv \\
&\leq \frac{d_8}{L_{12}+4}\int_1^{L_1-1}\int_{2}^{L_{12}-2} \ln\left(\frac{L_{12}-u+2}{u+2}\right)\frac{du}{L_{12}-2u}dv \\
&\leq \frac{d_8}{L_{12}+4}\int_1^{L_1-1}\int_{-\frac{L_{12}}{2}+2}^{\frac{L_{12}}{2}-2} \ln\left(\frac{L_{12}+4+2t}{L_{12}+4-2t}\right)\frac{dt}{2t}dv \\
&= \frac{d_8}{L_{12}+4}\int_1^{L_1-1}\int_{0}^{\frac{L_{12}}{4}-1} \ln\left(\frac{1+u}{u}\right)\frac{du}{2+u}dv \\
&\leq \frac{d_8}{L_{12}+4}\int_1^{L_1-1}\int_{0}^{+\infty} \ln\left(\frac{1+u}{u}\right)\frac{du}{2+u}dv \\
&\leq \frac{d_8\left(L_1-2\right)}{L_{12}+4}.
\end{aligned}$$

Finally

$$\beta_2 \leq c_7 \sum_L \left(\sum_{j-\epsilon^{02}k=L} \alpha_j^2 \, |j|^2 \, \alpha_k^2 |k|^2 \right)$$

$$= c_7 \left(\sum_j \alpha_j^2 \, |j|^2 \right)^2 . \qquad \blacksquare$$

Remarque 3.3.4 *To estimate β_3, it suffices to note that $\beta_3 \leq \beta_2$.*
Indeed

$$\begin{aligned}
\beta_3 &= \sum_{j+\epsilon^{02}k=\epsilon^{01}l+\epsilon^{03}m} \alpha_j \alpha_k \alpha_l \alpha_m \\
&= \sum_{\substack{l,m/l^i \geq 2-m^i \ (i=1,2) \\ l^i, m^i \in \mathbb{N}^* \ (i=3,4)}} \alpha_l \alpha_m \sum_{j+\epsilon^{02}k=\epsilon^{01}l+\epsilon^{03}m} \alpha_j \alpha_k \\
&= \sum_r \sum_m \alpha_m \alpha_{\epsilon^{01}R-\epsilon^{02}m} \sum_{j+\epsilon^{02}k=R} \alpha_j \alpha_k \\
&\leq \left(\sum_r \left(\sum_m \alpha_m \alpha_{\epsilon^{01}R-\epsilon^{02}m} \right)^2 \right)^{\frac{1}{2}} \left(\sum_r \left(\sum_{j+\epsilon^{02}k=R} \alpha_j \alpha_k \right)^2 \right)^{\frac{1}{2}} \\
&\leq \left(\sum_{m,l,R} \alpha_m \alpha_{\epsilon^{01}R-\epsilon^{02}m} \alpha_l \alpha_{\epsilon^{01}R-\epsilon^{02}l} \right)^{\frac{1}{2}} \left(\sum_r \left(\sum_{j+\epsilon^{02}k=R} \alpha_j \alpha_k \right)^2 \right)^{\frac{1}{2}} . \blacksquare
\end{aligned}$$

Note that

$$-\epsilon^{02}m + \left(\epsilon^{01}R - \epsilon^{02}l\right) = -\epsilon^{02}l + \left(\epsilon^{01}R - \epsilon^{02}m\right).$$

Then

$$\begin{aligned}
\beta_3 &\leq \left(\sum_{j+\epsilon^{02}k=l+\epsilon^{02}m} \alpha_j \alpha_k \alpha_l \alpha_m \right)^{\frac{1}{2}} \left(\sum_r \left(\sum_{j+\epsilon^{02}k=R} \alpha_j \alpha_k \right)^2 \right)^{\frac{1}{2}} \\
&= \sum_{j+\epsilon^{02}k=l+\epsilon^{02}m} \alpha_j \alpha_k \alpha_l \alpha_m \\
&= \beta_2. \qquad \blacksquare
\end{aligned}$$

3.3 Estimate of $\beta\left(\epsilon^1,\epsilon^2,\epsilon^3,\epsilon^4\right)$

Lemme 3.3.5 *There exists* $c_8>0$ *such that*

$$\beta\left(\epsilon^1,\epsilon^2,\epsilon^3,\epsilon^4\right)\leq c_8\left(\sum_j \alpha_j^2\,|j|^2\right)^2.$$

Proof - The constraint

$$\epsilon^1 j^1+\epsilon^2 j^2+\epsilon^3 j^3+\epsilon^4 j^4=0$$

will be written as

$$\sum_{i=1}^{2}\eta_j^i j_j^i=\sum_{i=3}^{4}\eta_j^i j_j^i \text{ for } j=1,2,3,4$$

where

$$\eta_j^i\in\{-1,1\} \text{ for } (i,j)\in\{1,2,3,4\}^2.$$

Let

$$K_j=\sum_{i=3}^{4}\eta_j^i j_j^i \text{ for } j\in\{1,2,3,4\}.$$

Then

$$j_j^4=\eta_j^4 K_j-\eta_j^4\eta_j^3 j_j^3.$$

Consequently,

$$\beta\left(\epsilon^1,\epsilon^2,\epsilon^3,\epsilon^4\right)=\sum_{j^3,K}\alpha_{j^3}\alpha_{\eta^4K-\eta^4\eta^3j^3}\sum_{\eta^1j^1+\eta^2j^2=K}\alpha_{j^1}\alpha_{j^2}$$

where $\eta=(\eta_1,\eta_2,\eta_3,\eta_4)$ and $K=(K_1,K_2,K_3,K_4)$.
It immediately follows that

$$\beta(\epsilon^1,\epsilon^2,\epsilon^3,\epsilon^4)$$

$$\leq\left(\sum_{j^3,\,l^3,\,K}\alpha_{j^3}\alpha_{\eta^4K-\eta^4\eta^3j^3}\alpha_{l^3}\alpha_{\eta^4K-\eta^4\eta^3l^3}\right)^{\frac12}\left(\sum_K\left(\sum_{\eta^1j^1\,+\,\eta^2j^2\,=\,K}\alpha_{j^1}\alpha_{j^2}\right)^2\right)^{\frac12}$$

$$\leq\left(\sum_{l^1+\eta^3\eta^4l^2=l^1+\eta^3\eta^4l^2}\alpha_{l^1}\alpha_{l^2}\alpha_{l^3}\alpha_{l^4}\right)^{\frac12}\left(\sum_K\left(\sum_{\eta^1j^1\,+\,\eta^2j^2\,=\,K}\alpha_{j^1}\alpha_{j^2}\right)^2\right)^{\frac12}.$$

Lemmas 3.3.1, 3.3.3, 3.3.4 and remark 3.3.4 may be summarized by saying that

$$\beta\left(\epsilon^1, \epsilon^2, \epsilon^3, \epsilon^4\right) \leq c_8 \left(\sum_j \alpha_j^2 \left|j\right|^2\right)^2$$

which is sufficient to conclude that

$$H_0^1\left(]0,1[^4\right) \subset L^4\left(]0,1[^4\right). \qquad \blacksquare$$

4.4 The case : $n = 3$

Note that the estimate of $\beta\left(\epsilon,\epsilon,\epsilon,-\epsilon,-\epsilon,-\epsilon\right)$ for each $\epsilon \in \{-1,1\}^3$ will be given in the same way as in lemma 2.2.1. The only cases that we will treat completely here are

- $\beta_4 = \sum_{j^1+\epsilon j^2+\epsilon j^3 = j^4+\epsilon j^5+\epsilon j^6} \alpha_{j^1}\alpha_{j^2}\alpha_{j^3}\alpha_{j^4}\alpha_{j^5}\alpha_{j^6},$
- $\beta_5 = \sum_{j^1+\epsilon j^2+\dot{\epsilon} j^3 = j^4+\epsilon j^5+\dot{\epsilon} j^6} \alpha_{j^1}\alpha_{j^2}\alpha_{j^3}\alpha_{j^4}\alpha_{j^5}\alpha_{j^6},$
- $\beta_6 = \sum_{j^1+j^2+\epsilon j^3 = j^4-\dot{\epsilon} j^5+\ddot{\epsilon} j^6} \alpha_{j^1}\alpha_{j^2}\alpha_{j^3}\alpha_{j^4}\alpha_{j^5}\alpha_{j^6}$

where

$$\epsilon = (-1,-1,1), \ \dot{\epsilon} = (-1,1,-1) \text{ and } \ddot{\epsilon} = (-1,1,1).$$

After that, we conclude the estimate of

$$\beta\left(\epsilon^1, \epsilon^2, \epsilon^3, \epsilon^4, \epsilon^5, \epsilon^6\right) = \sum_{\epsilon^1 j^1 + \cdots + \epsilon^6 j^6 = 0} \alpha_{j^1}\alpha_{j^2}\alpha_{j^3}\alpha_{j^4}\alpha_{j^5}\alpha_{j^6}$$

which will be sufficient to assert that

$$H_0^1\left(]0,1[^3\right) \subset L^6\left(]0,1[^3\right).$$

For any $S = (S_1, S_2, S_3) \in \mathbb{Z}^3$, denote $\hat{S}_1 = (S_2, S_3)$, $\hat{S}_2 = (S_1, S_3)$ and $\hat{S}_3 = (S_1, S_2)$.

4.1 Estimate of β_4

β_4 can be rewritten also in the following way

$$\begin{aligned}\beta_4 &= \sum_{j^1-\epsilon j^2-\epsilon j^3=j^4-\epsilon j^5-\epsilon j^6} \alpha_{j^1}\alpha_{j^2}\alpha_{j^3}\alpha_{j^4}\alpha_{j^5}\alpha_{j^6} \\ &= \sum_S \left(\sum_{\hat{j}_3+\hat{k}_3+\hat{l}_3=\hat{S}_3} \sum_{j_3-k_3-l_3=S_3} \alpha_j\alpha_k\alpha_l \right)^2 .\end{aligned}$$

Lemme 4.4.1 *There exists a positive constant c_9 such that*

$$\beta_4 \leq c_9 \left(\sum_j \alpha_j^2 \left|j\right|^2 \right)^3 . \tag{4.4.1}$$

Proof - Using Cauchy-Schwarz inequality, we have

$$\beta_4 \leq \sum_S \left(ds \sum_{\hat{j}_3+\hat{k}_3+\hat{l}_3=\hat{S}_3} \sum_{j_3-k_3-l_3=S_3} \alpha_j^2 \left|j\right|^2 \alpha_k^2 |k|^2 \alpha_l^2 |l|^2 \right) \times \varphi\left(S\right)$$

where

$$\varphi\left(S\right) = \sum_{\hat{j}_3+\hat{k}_3+\hat{l}_3=\hat{S}_3} \sum_{j_3-k_3-l_3=S_3} \frac{1}{\left|j\right|^2 |k|^2 |l|^2}.$$

But

$$\begin{aligned}\sum_{j_3-k_3-l_3=S_3} \frac{1}{\left|j\right|^2 |k|^2 |l|^2} &= \sum_{j_3-k_3-l_3=S_3} \frac{1}{\left(\left|\hat{j}_3\right|^2 + j_3^2\right)\left(\left|\hat{k}_3\right|^2 + k_3^2\right)\left(\left|\hat{l}_3\right|^2 + l_3^2\right)} \\ &\leq e_1 \sum_{j_3=1}^{+\infty} \frac{1}{\left(\left|\hat{j}_3\right|^2 + j_3^2\right)\left(\left|\hat{k}_3\right|^2 + j_3^2\right)\left(\left|\hat{l}_3\right|^2 + j_3^2\right)} \\ &\leq e_2 \int_1^{+\infty} \frac{dx}{\left(\left|\hat{j}_3\right|^2 + x^2\right)\left(\left|\hat{k}_3\right|^2 + x^2\right)\left(\left|\hat{l}_3\right|^2 + x^2\right)} \\ &\leq e_3 \kappa\left(\left|\hat{j}_3\right|, \left|\hat{k}_3\right|, \left|\hat{l}_3\right|\right)\end{aligned}$$

where

$$\kappa(x,y,z)=\begin{cases}\dfrac{1}{(y^2-x^2)(z^2-x^2)(x+1)}+\dfrac{1}{(x^2-y^2)(z^2-y^2)(y+1)}+\\ \dfrac{1}{(x^2-z^2)(y^2-z^2)(z+1)}\text{ if } x\neq y\neq z\neq x \text{ or}\\ \frac{-1}{(Y^2-X^2)^2(X+1)}+\frac{1}{3(Y^2-X^2)(X+1)^3}+\frac{1}{(Y^2-X^2)^2(Y+1)}\\ \text{with } X\neq Y \text{ and } (X,Y)\in\{(x=y,z),(x=z,y),(y=z,x)\} \text{ or}\\ \dfrac{1}{(x+1)^5}\text{ if } x=y=z.\end{cases}$$

Thus,

$$\sum_{j-\epsilon k-\epsilon l=S}\frac{1}{|j|^2|k|^2|l|^2}\leq e_4\Bigg(\sum_{\substack{\hat{j}_3+\hat{k}_3+\hat{l}_3=\hat{S}_3\\ |\hat{j}_3|\neq|\hat{k}_3|\neq|\hat{l}_3|\neq|\hat{j}_3|}}\kappa\left(|\hat{j}_3|,|\hat{k}_3|,|\hat{l}_3|\right)$$

$$+\sum_{\substack{\hat{j}_3+\hat{k}_3+\hat{l}_3=\hat{S}_3\\ |\hat{j}_3|=|\hat{k}_3|\neq|\hat{l}_3|}}\kappa\left(|\hat{j}_3|,|\hat{j}_3|,|\hat{l}_3|\right)+\sum_{\substack{\hat{j}_3+\hat{k}_3+\hat{l}_3=\hat{S}_3\\ |\hat{j}_3|=|\hat{l}_3|\neq|\hat{k}_3|}}\kappa\left(|\hat{j}_3|,|\hat{k}_3|,|\hat{j}_3|\right)$$

$$+\sum_{\substack{\hat{j}_3+\hat{k}_3+\hat{l}_3=\hat{S}_3\\ |\hat{k}_3|=|\hat{l}_3|\neq|\hat{j}_3|}}\kappa\left(|\hat{j}_3|,|\hat{k}_3|,|\hat{k}_3|\right)+\sum_{\substack{\hat{j}_3+\hat{k}_3+\hat{l}_3=\hat{S}_3\\ |\hat{k}_3|=|\hat{l}_3|=|\hat{j}_3|}}\kappa\left(|\hat{j}_3|,|\hat{j}_3|,|\hat{j}_3|\right)\Bigg).$$

We claim that

$$\sum_{\hat{j}_3+\hat{k}_3+\hat{l}_3=\hat{S}_3}\kappa\left(|\hat{j}_3|,|\hat{k}_3|,|\hat{l}_3|\right)\leq c_9$$

where c_9 is being independent of S. Indeed

$$\begin{aligned}&\sum_{|\hat{j}_3|\neq|\hat{k}_3|\neq|\hat{l}_3|\neq|\hat{j}_3|}\kappa\left(|\hat{j}_3|,|\hat{k}_3|,|\hat{l}_3|\right)\\ &\leq\sum_{|\hat{j}_3|\neq|\hat{k}_3|\neq|\hat{l}_3|\neq|\hat{j}_3|}\frac{1}{2\left(|\hat{j}_3|+1\right)\left(|\hat{k}_3|+1\right)\left(|\hat{l}_3|+1\right)}\\ &\leq e_5\sum_{j_1=1}^{S_1}\sum_{j_2=1}^{S_2}\frac{1}{\left(|\hat{j}_3|+1\right)^3}\\ &\leq c_9.\end{aligned}$$

Let a and $b \in I\!N^*$ such that

$$a \neq b \text{ and } (a,b) \in \left\{ \left(|\hat{j}_3| = |\hat{k}_3|, |\hat{l}_3| \right), \left(|\hat{j}_3| = |\hat{l}_3|, |\hat{k}_3| \right), \left(|\hat{k}_3| = |\hat{l}_3|, |\hat{j}_3| \right) \right\}.$$

Assume, for example, that $a = |\hat{j}_3| = |\hat{k}_3|$ and $b = |\hat{l}_3|$, then

$$\begin{aligned}
\sum_{\hat{j}_3 + \hat{k}_3 + \hat{l}_3 = \hat{s}_3} \kappa(a,a,b) &= \sum_{a \neq b} \frac{3a^2 + (5-b)a + \left(3 - b - b^2\right)}{3(a+b)(a^2-b^2)(b+1)(a+1)^3} \\
&= \sum_{a > b} \frac{3a^2 + (5-b)a + \left(3 - b - b^2\right)}{3(a+b)(a^2-b^2)(b+1)(a+1)^3} \\
&+ \sum_{a < b} \frac{-3a^2 - (5-b)a - \left(3 - b - b^2\right)}{3(a+b)(b^2-a^2)(b+1)(a+1)^3} \\
&\leq \sum_{a > b} \frac{3a^2 + 6a + 3}{3(a+b)(a^2-b^2)(b+1)(a+1)^3} \\
&+ \sum_{a < b} \frac{2b^2 + b - 3}{3(a+b)(b^2-a^2)(b+1)(a+1)^3} \\
&\leq \sum_{b} \frac{1}{(b+1)^3} + \sum_{a} \frac{1}{(a+1)^3} \\
&= 2\sum_{a} \frac{1}{(a+1)^3} \\
&\leq c_9.
\end{aligned}$$

It is easy to prove that

$$\sum_{\substack{\hat{j}_3 + \hat{k}_3 + \hat{l}_3 = \hat{s}_3 \\ |\hat{j}_3| = |\hat{k}_3| = |\hat{l}_3|}} \frac{1}{\left(|\hat{j}_3| + 1\right)^5} \leq c_9.$$

Next we obtain

$$\begin{aligned}
\beta_4 &\leq c_9 \sum_{S} \left(\sum_{j - \epsilon k - \epsilon l = S} \alpha_j^2 |j|^2 \alpha_k^2 |k|^2 \alpha_l^2 |l|^2 \right) \\
&\leq c_9 \left(\sum_{j} \alpha_j^2 |j|^2 \right)^3. \qquad \blacksquare
\end{aligned}$$

4.2 Estimate of β_5

Lemme 4.4.2 *There exists a constant* $c_{10} > 0$ *such that*

$$\beta_5 \leq c_{10} \left(\sum_j \alpha_j^2 \left| j \right|^2 \right)^3 . \tag{4.4.2}$$

Proof - It suffices to prove that

$$\tilde{\beta}_5 = \sum_{j + \dot{\epsilon} k + \ddot{\epsilon} l = S} \frac{1}{\left| j \right|^2 |k|^2 |l|^2} \leq c_{10}.$$

We may proceed exactly as in the proof of lemma 4.4.1 and conclude that

$$\tilde{\beta}_5 \leq e_8 \kappa \left(j_1, k_1, l_1 \right)$$

where

$$\kappa \left(x, y, z \right) = \begin{cases} \displaystyle\sum_{(X,Y,Z) \in \aleph} \frac{\ln \left(X + 2 \right)}{\left(\left(Y + 2 \right)^2 - \left(X + 2 \right)^2 \right) \left(\left(X + 2 \right)^2 - \left(Z + 2 \right)^2 \right)} \\ \text{with } X \neq Y \neq Z \neq X \text{ and } \aleph = \left\{ \left(x, y, z \right), \left(y, z, x \right), \left(z, x, y \right) \right\} \\ \text{or} \\ \left(\frac{(Y+2)^2 - (X+2)^2}{(X+2)^2} - \ln \frac{(Y+2)^2}{(X+2)^2} \right) \frac{1}{2\left((Y+2)^2 - (X+2)^2 \right)^2} \\ \text{with } X \neq Y \text{ and } (X,Y) \in \left\{ \left(x = y, z \right), \left(x = z, y \right), \left(y = z, x \right) \right\} \\ \text{or} \\ \dfrac{1}{\left(x + 1 \right)^2} \text{ if } x = y = z. \end{cases}$$

An easy computation shows that

$$\sum_{a<b} \left(\frac{b^2 - a^2}{a^2} - \ln \left(\frac{b^2}{a^2} \right) \right) \frac{1}{2 \left(b^2 - a^2 \right)^2} \leq \sum_{a<b} \frac{1}{4b^3}$$

and

$$\sum_{a>b} \left(\frac{b^2 - a^2}{a^2} - \ln \left(\frac{b^2}{a^2} \right) \right) \frac{1}{2 \left(b^2 - a^2 \right)^2} \leq \sum_{a>b} \frac{1}{4a^3}.$$

Assume that a, b and c are three any integers satisfying $a-b-c=S_1+1$, then a simple verification shows that

$$\begin{aligned}\chi(S_1) &= \sum_{a\neq b\neq c\neq a}\frac{\left(c^2-b^2\right)\ln a^2+\left(a^2-c^2\right)\ln b^2+\left(b^2-a^2\right)\ln c^2}{2\left(c^2-b^2\right)\left(a^2-c^2\right)\left(b^2-a^2\right)}\\ &\leq 6\sum_{a>b>c}\frac{\ln b}{\left(a^2-b^2\right)\left(b^2-c^2\right)}\\ &\leq \tfrac{3}{2}\sum_{a>b}\frac{\ln b}{ab}.\end{aligned}$$

Using the fact that $\lim\limits_{c\to+\infty}\dfrac{\ln c}{\sqrt{c}}=0$, we conclude that there exists $A>0$ such that

$$\chi(S_1)\leq\sum_{a=1}^{+\infty}\frac{\ln A}{a^2}+\sum_{a=1}^{+\infty}\frac{1}{a\sqrt{a}}.$$

This brings the proof to its final stage. ■

Lemme 4.4.3 *There exists c_{13} such that*

$$\beta_6\leq c_{13}\left(\sum_j\alpha_j^2\left|j\right|^2\right)^3. \tag{4.4.3}$$

Proof - A similar analysis as in the proof of remark 3.3.4 shows that

$$\beta_6\leq\beta_4^{\frac{1}{2}}\beta_5^{\frac{1}{2}}$$

which terminates the proof. ■

Remarque 4.4.1 *Neither the hypothesis nor the conclusion is affected if we replace $n\geq 3$ by $n=3$ in lemma 2.2.2 and* (2.2.12) *by lemmas 4.4.2, 4.4.3 and remark 4.4.1.*

Our next corollary is an adaptation of the lemma 2.2.2

Corollaire 4.4.1 *For $i\in\{1,\ldots,6\}$, let $\epsilon^i\in\{-1,1\}^3$. There exists $c_{14}>0$ such that*

$$\sum_{\epsilon^1j^1+\cdots+\epsilon^6j^6=0}\prod_{i=1}^{6}\alpha_{j^i}\leq c_{14}\left(\sum_j\alpha_j^2\left|j\right|^2\right)^3. \tag{4.4.4}$$

5.5 Some embeddings in the case $n = 2$

In much the same way as in section 2.2, we extend our method to study the injection

$$H^{1-\frac{1}{p}}(\Omega) \subset L^{2p}(\Omega) \text{ for all } q \in I\!N^*. \tag{5.5.1}$$

In this context we give the number of the series $\beta\left(\epsilon^1, \epsilon^2, \ldots, \epsilon^{2p}\right)$ that we have to treat. Indeed, the constraint

$$\sum_{i=1}^{2p} \epsilon^i j^i = 0$$

can be written without loss of generality

$$\begin{cases} j_1^1 + j_1^2 + \cdots + j_1^{p+k} &= j_1^{p+k+1} + \cdots + j_1^{2p} \\ j_2^{\sigma(1)} + j_2^{\sigma(2)} + \cdots + j_2^{\sigma(p+k+r2)} &= j_2^{\sigma(p+k+r+1)} + \cdots + j_2^{\sigma(2p)} \end{cases} \quad \left(\tilde{C}\right)$$

where σ is permutation of $\{1, 2, \ldots, 2p\}$, $k \in \{0, 1, \ldots, p-1\}$, $r \in \{0, 1, \ldots, p-k-1\}$ and $j^i = \left(j_1^i, j_2^i\right)$ for $i \in \{0, 1, \ldots, 2p\}$. A slight change in the proof of lemma 2.2.2 shows that if

$$\sum_{\left(\tilde{C_0}\right)} \prod_{i=1}^{2p} \alpha_{j^i} \leq c \left(\sum_j \alpha_j^2 \, |j|^{\frac{2(p-1)}{p}} \right)^p, \tag{5.5.2}$$

then

$$\sum_{\left(\tilde{C}\right)} \prod_{i=1}^{2p} \alpha_{j^i} \leq c \left(\sum_j \alpha_j^2 \, |j|^{\frac{2(p-1)}{p}} \right)^p, \tag{5.5.3}$$

where $\left(\tilde{C_0}\right)$ is the constraint $(\tilde{C})$ in which there $k = r = 0$. It is now known that (5.5.3) implies

$$H^{1-\frac{1}{p}}(\Omega) \subset L^{2p}(\Omega).$$

The number of the series, like the left hand side of inequality (5.5.2), that we have to consider is

$$\left[\frac{p}{2}\right] + 1 \tag{5.5.4}$$

which is computed when σ is in the set of permutations of $\{1, 2, \ldots, 2p\}$.

1. The case $p = 2$.
   As we have indicated, to prove $H^{\frac{1}{2}}(\Omega) \subset L^4(\Omega)$, we have to estimate

$$\beta(\epsilon, \epsilon, -\epsilon, -\epsilon) \text{ and } \beta(\epsilon, \epsilon', -\epsilon, -\epsilon')$$

   where

$$\epsilon, \epsilon' \in \{-1, 1\}^2, \; \epsilon' \neq \epsilon \text{ and } \epsilon' \neq -\epsilon.$$

   The countrepart of lemma 3.3.1 is

   Lemme 5.5.1 *There exists $c_{10} > 0$ such that for any $\epsilon \in \{-1, 1\}^2$*

$$\beta(\epsilon, \epsilon, -\epsilon, -\epsilon) \leq c_{10} \left(\sum_j \alpha_j^2 |j| \right)^2. \tag{5.5.5}$$

   To verify that

$$\beta(\epsilon, \epsilon', -\epsilon, -\epsilon') \leq c \left(\sum_j \alpha_j^2 |j| \right)^2,$$

   we have to prove that

$$\sum_{\epsilon j + \epsilon' k = L} \frac{1}{|j| |k|} \leq c$$

   where c is being independent of L. Indeed

$$\begin{aligned} \sum_{\epsilon j + \epsilon' k = L} \frac{1}{|j| |k|} &\leq a_1 \sum_{j_2 + k_2 = L_2} \sum_{j_1 = 1}^{+\infty} \frac{1}{(j_1 + j_2)(j_1 + k_2)} \\ &\leq a_2 \sum_{j_2 + k_2 = L_2} \kappa(j_2, k_2) \end{aligned}$$

   where

$$\kappa(x, y) = \begin{cases} \dfrac{1}{y - x} \ln\left(\dfrac{y + 1}{x + 1} \right) & \text{if } \; x \neq y \\ \dfrac{1}{1 + x} & \text{if } \; x = y. \end{cases}$$

Then,

$$\sum_{\epsilon j + \epsilon' k = L} \frac{1}{|j|\,|k|}$$

$$\leq a_2 \left(\sum_{j_2 = 1/2 j_2 \neq L_2}^{L_2 - 1} \frac{1}{L_2 - 2j_2} \ln \left(\frac{L_2 - j_2 + 1}{j_2 + 1} \right) + \frac{(-1)^{L_2} + 1}{2} \frac{2}{L_2 + 2} \right)$$

$$\leq a_3 \left(\tfrac{1}{2} + \int_0^{\frac{L_2 - 1}{2}} \frac{\ln(1+u)}{u} \frac{du}{2+u} \right)$$

$$\leq c,$$

which is our claim. ∎

2. The case $p = 3$.
   We are concerned with the injection

$$H^{\frac{2}{3}}(\Omega) \subset L^6(\Omega). \tag{5.5.6}$$

To prove (5.5.6), (5.5.4) suggests that we have to treat

$$\beta(\epsilon, \epsilon, \epsilon, -\epsilon - \epsilon, -\epsilon) \text{ and } \beta\left(\epsilon, \epsilon, \epsilon', -\epsilon, -\epsilon, -\epsilon'\right)$$

where

$$\epsilon, \epsilon' \in \{-1, 1\}^2,\ \epsilon' \neq \epsilon \text{ and } \epsilon' \neq -\epsilon.$$

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Received: March, 2009